Group Theory and Computational Methods
ICTS-TIFR, Bangalore, 05 – 14 November 2016
1. Condensation
2. An Example: The Fischer Group $Fi_{23}$ Modulo 2
Throughout this lecture, $G$ denotes a finite group and $F$ a field.

Also, $\mathcal{A}$ denotes a finite-dimensional $F$-algebra, $J(\mathcal{A})$ the Jacobson radical of $\mathcal{A}$ (i.e., the intersection of the maximal right ideals of $\mathcal{A}$).

$\text{mod-}\mathcal{A}$: category of finite-dimensional right $\mathcal{A}$-modules
The MeatAxe can reduce representations of degree up to 200,000 over $\mathbb{F}_2$.

Over larger fields, only smaller degrees are feasible.

To overcome this problem, Condensation is used (Thackray, Parker, ca. 1980).
Let $e \in \mathcal{A}$ an idempotent, i.e., $0 \neq e = e^2$ (a projection).

Get exact functor: $\text{mod-} \mathcal{A} \rightarrow \text{mod-} e\mathcal{A}e$, $V \mapsto Ve$.

If $S \in \text{mod-} \mathcal{A}$ is simple, then $Se = 0$ or simple.

Let $S_1, \ldots, S_n$ be the simple $\mathcal{A}$-modules (up to isomorphism).

Suppose that $S_1 e \neq 0, \ldots, S_m e \neq 0$, $S_{m+1} e = \cdots = S_n e = 0$.

Then $S_1 e, \ldots, S_m e$ are exactly the simple $e\mathcal{A}e$-modules (up to isomorphism).

As the Condensation functor is exact, it sends a composition series of $V \in \text{mod-} \mathcal{A}$ to a composition series of $Ve \in \text{mod-} e\mathcal{A}e$. 
If \( S \neq 0 \) for all simple \( S \in \text{mod-} \mathcal{A} \), then Condensation is an equivalence of categories, i.e. \( \mathcal{A} \) and \( e \mathcal{A} e \) are Morita equivalent.

An indecomposable direct summand of \( \mathcal{A} \mathcal{A} \) is called a PIM.

(A PIM in the sense of Lecture 2 is the Brauer character of a PIM of \( FG \), extended by 0 from \( G_{p'} \) to \( G \).

A projective \( \mathcal{A} \)-module is a direct sum of PIMs.

A finite-dimensional \( F \)-algebra \( \mathcal{B} \) is Morita equivalent to \( \mathcal{A} \), if \( \mathcal{B} \cong \text{End}_{\mathcal{A}}(Q) \) for a projective module \( Q \) of \( \mathcal{A} \) containing every PIM of \( \mathcal{A} \) (up to isomorphism) as a direct summand.

Morita equivalent algebras have “the same” representations.
Let $H \leq G$ with $\text{char}(F) \nmid |H|$. Then

$$e := e_H := \frac{1}{|H|} \sum_{x \in H} x \in FG$$

is a suitable idempotent.

Other idempotents can be used, e.g.,

$$e = \frac{1}{|H|} \sum_{x \in H} \lambda(x^{-1}) x \in FG,$$

where $\lambda : H \to F^*$ is a homomorphism (Noeske, 2005).
Let $e := e_H = 1/|H| \sum_{x \in H}$ be as above.

Let $V$ be the permutation $FG$-module w.r.t. an action of $G$ on the finite set $\Omega$. Then $Ve$ is the set of $H$-fixed points in $V$.

**Task:** Given $g \in G$, determine the action of $ege$ on $Ve$, without the explicit computation of the action of $g$ on $V$.

**Theorem (Thackray and Parker, 1981)**

*This can be done!*
Let $\Omega_1, \ldots, \Omega_m$ be the $H$-orbits on $\Omega$.

The orbits sums $\hat{\Omega}_j := \sum_{\omega \in \Omega_j} \omega \in V$ form a basis of $Ve$.

W.r.t. this basis, the $(i, j)$-entry $a_{ij}$ of the matrix of $ege$ on $Ve$ equals

$$a_{ij} = \frac{1}{|\Omega_j|} |\Omega_i g \cap \Omega_j|.$$
To perform these computations, we need to be able to

1. compute $\Omega_j$ and $|\Omega_j|$, $1 \leq j \leq m$,
2. decide $\omega \in \Omega_j$? for given $\omega \in \Omega$ and $1 \leq j \leq m$.

In actual applications, $|\Omega| \approx 10^{15}$, so the elements of $\Omega$ can not be stored in memory.

Parker and Wilson suggested Direct Condensation methods; these were later extended and implemented by Cooperman, Lübeck, Müller and Neunhöffer.

**Principal idea:** Enumerate the $H$-orbits $\Omega_j$ by suborbits of subgroups $U \leq H$. Iterate this idea.

Details depend on the realisation of the action on $\Omega$. 
Let $V$ and $W$ be two $FG$-modules.

**Task:** Given $g \in G$, determine the action of $ege$ on $(V \otimes W)e$, without the explicit computation of the action of $g$ on $V \otimes W$.

**Theorem (Lux and Wiegelmann, 1997)**

*This can be done!*

Let $M$ be a subgroup of $G$ and let $W$ be an $FM$-module.

The **induced module** is the $FG$-module $W \otimes_{FM} FG$.

**Task:** Given $g \in G$, determine action of $ege$ on $(W \otimes_{FM} FG)e$, without the explicit computation of the action of $g$ on $W \otimes_{FM} FG$.

**Theorem (Müller and Rosenboom, 1997)**

*This can be done!*
Then $\text{Hom}_{FM}(V, W)$ is a right $FG$-module:

$$v(\varphi g) := (gv)\varphi, \quad v \in V, \varphi \in \text{Hom}_{FM}(V, W), g \in G.$$ 

**Examples**

1. $\text{Hom}_{FM}(FG, F) \cong \text{permutation module corresponding to permutation action of } G \text{ on } \Omega := M \setminus G.$

2. $\text{Hom}_F(V^*, W) \cong V \otimes W \text{ for } V, W \in \text{mod}-FG.$

3. $\text{Hom}_{FM}(FG, W) \cong W \otimes_{FM} FG.$

Lux, Neunhöffer, Noeske develop general Condensation programs for such homomorphism spaces.
**CONDENSATION: SOME APPLICATIONS**

Benson, Conway, Parker, Thackray, Thompson, 1980:
Existence of $J_4$.

Thackray, 1981:
2-modular character table of McL.
Answer to a question of Brauer.

Cooperman, H., Lux, Müller, 1997:
Brauer tree of Th modulo 19.
dim($V$) = 976 841 775, dim($V_e$) = 1403.

Müller, Neunhöffer, Röhr, Wilson, 2002:
Brauer trees of Ly modulo 37 and 67.
dim($V$) = 1 113 229 656.

More applications later.
A finite-dimensional $F$-algebra $\mathcal{B}$ is called basic, if

$$\mathcal{B} \cong Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$$

with PIMs $Q_i$ such that $Q_i \not\cong Q_j$ for $1 \leq i \neq j \leq n$.

Alternatively, if $\mathcal{B} / J(\mathcal{B})$ is a direct sum of division algebras.

**Facts**

Let $P_1, \ldots, P_n$ be the PIMs of $\mathcal{A}$ (up to isomorphism). Then

$$\mathcal{B} := \text{End}_\mathcal{A}(P_1 \oplus \cdots \oplus P_n)$$

is a basic algebra Morita equivalent to $\mathcal{A}$, the basic algebra of $\mathcal{A}$.

This is the smallest algebra Morita equivalent to $\mathcal{A}$.
If \( \dim(\mathcal{A}) \) is large, it may be too difficult to construct the basic algebra of \( \mathcal{A} \) explicitly.

Klaus Lux uses Condensation to construct algebras of feasible dimensions, Morita equivalent to (blocks of) group algebras \( FG \).

Need idempotent \( e \in FG \) with \( Se \neq 0 \) for all simple \( FG \)-modules \( S \) (or all simple modules in a block).

This can be checked with the modular character table of \( G \), if \( e = e_H \) for some \( H \leq G \) with \( \text{char}(F) \nmid |H| \).

**Example:** Principal block \( \mathcal{B}_0 \) of \( HS \) modulo 5, \( |H| = 192 \).
\[
\dim(\mathcal{B}_0) = 15\,364\,500, \quad \dim(e_H\mathcal{B}_0e_H) = 767.
\]

See Klaus Lux, *Faithful Condensation for Sporadic Groups*, (http://math.arizona.edu/~klux/habil.html).

**Applications:** Cartan matrices for group algebras, cohomology computations.
Condensation: History

H ≤ G

Hecke $H \times H$ in $F_0$

multiply as in $F_0$

\[ \sigma_H = \text{image} \, \sigma(H \times H) \]

\[ \sigma_H(x \times y) = \sigma(H \times y \cdot H) \]

\[ \sigma_H(x \times y) = \sigma(H \times y \cdot H) \]

use this line to define $x$. 

An Example: The Fischer Group $F_{23}$ modulo 2
We investigate $Ve$ through the MeatAxe, using matrices of generators of $eFGe$.

**Question (The Generation Problem)**

How can $eFGe$ be generated with “a few” elements?

If $\mathcal{E} \subseteq FG$ with $F\langle \mathcal{E} \rangle = FG$, then in general $F\langle e\mathcal{E}e \rangle \nleq eFGe$.

- Let $\mathcal{C} := F\langle e\mathcal{E}e \rangle \leq eFGe$.
  Instead of $Ve$ we consider the $\mathcal{C}$-module $Ve|_{\mathcal{C}}$.

- We can draw conclusions on $V$ from $Ve$, but not from $Ve|_{\mathcal{C}}$. 
**Theorem (F. Noeske, 2005)**

Let $H \trianglelefteq N \leq G$. If $\mathcal{T}$ is a set of double coset representatives of $N \backslash G / N$ and $\mathcal{N}$ a set of generators of $N$, then we have for $e = e_H$:

\[ eFGe = F\langle e\mathcal{N}e, e\mathcal{T}e \rangle \]

as $F$-algebras.

More sophisticated results by Noeske on generation are available, but have not found applications yet.

**Matching Problem:** Let $e, e' \in FG$ be idempotents. Suppose $S, S' \in \text{mod-}FG$ are simple, and we know $Se$ and $S'e'$. Can we decide if $S \cong S'$? Yes! (Noeske, 2008)
Condensing Projective Modules

Not a new idea, but now feasible through

- improved Condensation techniques
- programs by Jon Carlson for matrix algebras (see next lecture)

If $P = eFG$ is projective, then $\text{End}_{FG}(P) = eFGe = Pe$, and “generation” can be checked. E.g. $\dim(\text{End}_{FG}(P))$ known.

**Example ($G = Th, \rho = 5$)**

1. *(Done in 2007 with Jon Carlson):*
   
   $P = e_HFG$, for $H = 3xG_2(3)$, $\dim(P) = 7124544000$, $\dim_F(\text{End}_{FG}(P)) = 788 \rightsquigarrow$ some progress

2. *(Envisaged):*
   
   $\dim(Q) = 43957879875$, $\dim_F(\text{End}_{FG}(Q)) = 21530 \rightsquigarrow$ almost finish Th modulo 5
Let $G$ denote the Fischer group $Fi_{23}$.

This is a sporadic simple group of order

$$4 089 470 473 293 004 800.$$ 

$G$ has a maximal subgroup $M$ of index 31 671, isomorphic to $2.Fi_{22}$, the double cover of the Fischer group $Fi_{22}$.

In joint work with Max Neunhöffer and F. Noeske we have computed the 2-modular character table of $G$.
In the following, let $F = \mathbb{F}_2$, the field with 2 elements.

Let $\Omega := M\backslash G$ and let $V$ denote the corresponding permutation module over $F$ (thus $\dim_F(V) = 31\,671$).

Using the MeatAxe we found: $V$ contains composition factors $1, 782, 1\,494, 3\,588, 19\,940$ (denoted by their degrees).
(This took about 4 days of CPU time in 8 GB main memory.)

Using Condensation we analysed the ten tensor products:

$$782 \otimes 782, 782 \otimes 1\,494, \ldots, 19\,940 \otimes 19\,940.$$ 

Note: $\dim_F(19\,940 \otimes 19\,940) = 367\,603\,600$.

One such matrix over $\mathbb{F}_2$ would need $\approx 18\,403\,938$ GB.
We took $H \leq G$, $|H| = 3^9 = 19\,683$.

We found that $eFGe$ and $FG$ are Morita equivalent (a posteriori).

$\dim_F (19\,940 \otimes 19\,940) e = 25\,542$.

One such matrix over $\mathbb{F}_2$ needs $\approx 77.8$ MB.

About 1 week of CPU time to compute the action of one element $ege$ on $(19\,940 \otimes 19\,940)e$.

Every irreducible $FG$-module (of the principal 2-block) occurs in $19\,940 \otimes 19\,940$. 
The results of the Condensation and further computations with Brauer characters using GAP and MOC gave all the irreducible 2-modular characters of $G$.

Degrees of the irreducible 2-modular characters of $Fi_{23}$:

\begin{align*}
1, & \quad 782, & \quad 1494, & \quad 3588, \\
19940, & \quad 57408, & \quad 79442, & \quad 94588, \\
94588, & \quad 583440, & \quad 724776, & \quad 979132, \\
1951872, & \quad 1997872, & \quad 1997872, & \quad 5812860, \\
7821240, & \quad 8280208, & \quad 17276520, & \quad 34744192, \\
73531392, & \quad 97976320, & \quad 166559744, & \quad 504627200, \\
& \quad 504627200.
\end{align*}

Using similar methods, Görgen and Lux have recently computed the irreducible characters of $Fi_{23}$ over $\mathbb{F}_3$. (Largest condensed module: 184 644, largest module found: 34 753 159.)
REFERENCES

Thank you for your attention!