In the following exercises, $G$ is a finite group and $F$ a field.

1. Let $N$ be a normal subgroup of $G$. Show that every (irreducible) $F$-representation of $G/N$ yields an (irreducible) $F$-representation of $G$.

2. Show that two $F$-representations of $G$ of degree 1 are equivalent, if and only if they are equal.

3. Let $G'$ denote the commutator subgroup of $G$. Prove that $G$ and $G/G'$ have the same number of irreducible $F$-representation of degree 1.

4. Compute (by hand) the ordinary character table of the symmetric group $S_3$.

5. Compute (by hand) the ordinary character table of the symmetric group $S_4$. Use the fact that $S_4$ has a factor group isomorphic to $S_3$. Construct an irreducible $\mathbb{C}$-representation of $S_4$ of degree 3 by finding an invariant subspace in a 4-dimensional $\mathbb{C}$-vector space $V$, on which $S_4$ acts by permuting four basis vectors $v_1, \ldots, v_4$ in the same way as it permutes the letters $1, \ldots, 4$.

6. Let $\chi$ and $\lambda$ be two $\mathbb{C}$-characters of $G$ with $\lambda(1) = 1$. Show that $\chi \cdot \lambda : G \to \mathbb{C}, g \mapsto \chi(g)\lambda(g)$ is a $\mathbb{C}$-character of $G$.

7. Let $\mathfrak{X}$ and $\mathfrak{Y}$ denote two $F$-representations of $G$ on the vector spaces $U$ and $W$, respectively. Show that there is an $F$-representation $\mathfrak{Z}$ of $G$ on $U \otimes_F W$ such that

$$(u \otimes w)\mathfrak{Z}(g) = u\mathfrak{X}(g) \otimes w\mathfrak{Y}(g)$$

for all $u \in U$, $w \in W$ and $g \in G$.

Compute the character $\chi_3$ from the characters $\chi_{\mathfrak{X}}$ and $\chi_{\mathfrak{Y}}$. 
8. Let $\chi$ be a $\mathbb{C}$-character of $G$ and let $g \in G$. Show that $\chi(g^{-1}) = \overline{\chi(g)}$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

9. Prove the Second Orthogonality Relations: Let $\text{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$ denote the set of irreducible $\mathbb{C}$-characters of $G$, and let $g, h \in G$. Show that

$$\sum_{i=1}^{k} \chi_i(g) \overline{\chi_i(h^{-1})} = \begin{cases} |C_G(g)|, & \text{if } g \text{ and } h \text{ are conjugate in } G \\ 0, & \text{otherwise} \end{cases}$$

Hint: Write the Orthogonality Relations as a matrix equation and use the fact that an invertible matrix commutes with its inverse.

10. Use the character table of the sporadic group $M_{11}$ to show that the alternating group $A_5$ is a subgroup of $M_{11}$.

Hint: Use GAP to compute structure constants of $M_{11}$ and a presentation of $A_5$, e.g. $A_5 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$.

11. Let $F$ be finite of characteristic $p$ and let $G$ be a $p$-group. Show that $G$ has a unique irreducible $F$-representation.

12. Use GAP to compute the $p$-modular character tables of the alternating group $A_5$ for $p = 3, 5$. What about $p = 2$? (The $p$-modular character table of $G$ is the Brauer character table of $G$ with respect to an algebraically closed field $F$ of characteristic $p$.)

Hint: Produce projective characters using suitable products of characters or induced characters. Use the fact that $|G|_p$, the $p$-part of $|G|$, divides $\Phi(1)$ for every PIM $\Phi$.

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**References**
