FINITE GROUPS OF LIE TYPE AND THEIR REPRESENTATIONS – LECTURE I

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CONTENTS

THE CLASSIFICATION OF THE FINITE SIMPLE GROUPS

"Most" finite simple groups are closely related to finite groups of Lie type.

THEOREM

Every finite simple group is

1. *one of 26 sporadic simple groups; or*
2. *a cyclic group of prime order; or*
3. *an alternating group A_n with $n \geq 5$; or*
4. *closely related to a finite group of Lie type.*

What are finite groups of Lie type?

Finite analogues of Lie groups.

THE FINITE CLASSICAL GROUPS

Examples for finite groups of Lie type are the finite classical groups.
These are classical groups, i.e. full linear groups or linear groups preserving a form of degree 2, defined over finite fields.

EXAMPLES

- $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{Sp}_{2m}(q)$, $\mathrm{SO}_{2m+1}(q)$ ... *(q a prime power)*
- *E.g.,* $\mathrm{SO}_{2m+1}(q) = \{g \in SL_{2m+1}(q) \mid g^{tr} Jg = J\}$, *with*
  $$J = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{F}_q^{2m+1 \times 2m+1}.$$
- *Related groups, e.g.,* $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$, $\mathrm{CSp}_{2m}(q)$ *etc. are also classical groups.*

Not all classical groups are simple, but closely related to simple goups, e.g. $\mathrm{SL}_n(q) \to \mathrm{PSL}_n(q) = \mathrm{SL}_n(q)/Z(SL_n(q))$.

EXCEPTIONAL GROUPS

There are groups of Lie type which are not classical, namely,

Exceptional groups: $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$ (q a prime power, the order of a finite field),

Twisted groups: ${}^2E_6(q)$, ${}^3D_4(q)$ (q a prime power),

Suzuki groups: ${}^2B_2(2^{2m+1})$ ($m \geq 0$),

Ree groups: ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$ ($m \geq 0$).

The names of these goups, e.g. $G_2(q)$ or $E_8(q)$ refer to simple complex Lie algebras or rather their root systems.

How are groups of Lie type constructed? What are their properties, subgroups, orders, etc?

THE ORDERS OF SOME FINITE GROUPS OF LIE TYPE

$$|\mathrm{GL}_n(q)| = q^{n(n-1)/2}(q-1)(q^2-1)(q^3-1)\cdots(q^n-1).$$

$$|\mathrm{GU}_n(q)| = q^{n(n-1)/2}(q+1)(q^2-1)(q^3+1)\cdots(q^n-(-1)^n).$$

$$|\mathrm{SO}_{2m+1}(q)| = q^{m^2}(q^2-1)(q^4-1)\cdots(q^{2m}-1).$$

$$|F_4(q)| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1).$$

$$|{}^2F_4(q)| = q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1) \quad (q = 2^{2m+1}).$$

Is there a systematic way to derive these order formulae?

ROOT SYSTEMS

We take a little detour to discuss root systems.
Let V be a finite-dimensional real vector space endowed with an inner product $(-,-)$.

DEFINITION

A root system in V is a finite subset $\Phi \subset V$ satisfying:

1. *Φ spans V as a vector space and $0 \notin \Phi$.*
2. *If $\alpha \in \Phi$, then $r\alpha \in \Phi$ for $r \in \mathbb{R}$, if and only if $r \in \{\pm 1\}$.*
3. *For $\alpha \in \Phi$ let s_α denote the reflection on the hyperspace orthogonal to α:*

   $$s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha, \quad v \in V.$$

   Then $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
4. *$2(\beta,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.*

WEYL GROUP AND DYNKIN DIAGRAM

Let Φ be a root system in the inner product space V.
The group

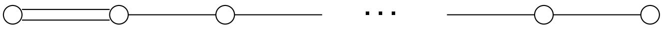
$$W := W(\Phi) := \langle s_\alpha \mid \alpha \in \Phi \rangle \leq O(V)$$

is called the Weyl group of Φ.
There is a subset $\Pi \leq \Phi$ such that

1. Π is a basis of V.
2. Every $\alpha \in \Phi$ is an integer linear combination of Π with either only non-negative or only non-positive coefficients.

Such a Π is called a base of Φ.
The Dynkin diagram of Φ is the graph with nodes $\alpha \in \Pi$, and $4(\alpha, \beta)^2/(\alpha, \alpha)(\beta, \beta)$ edges between the nodes α and β. E.g.

B_r:



CHEVALLEY GROUPS

Chevalley groups are (subgroups of) automorphism groups of finite classical Lie algebras.

Classical Lie algebra: A Lie algebra corresponding to a (finite-dimensional) simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.

These have been classified by Killing and Cartan (1890s) in terms of root systems.

Let Φ be the root system of $\mathfrak{g}$, and let Π be a base of Φ.

$\mathfrak{g}$ has a Chevalley basis $\mathcal{C} := \{e_r \mid r \in \Phi, h_r, r \in \Pi\}$, such that all structure constants w.r.t. $\mathcal{C}$ are integers.

Let $\mathfrak{g}_{\mathbb{Z}}$ denote the $\mathbb{Z}$-form of $\mathfrak{g}$ constructed from $\mathcal{C}$.

If k is a field, then $\mathfrak{g}_k := k \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$ is the classical Lie algebra corresponding to $\mathfrak{g}$.

CHEVALLEY'S CONSTRUCTION (1955)

Let $\mathfrak{g}$ be a (finite-dimensional) simple Lie algebra over $\mathbb{C}$ with Chevalley basis $\mathcal{C}$.

For $r \in \Phi$, $\zeta \in \mathbb{C}$, there are $x_r(\zeta) \in \mathrm{Aut}(\mathfrak{g})$ defined by

$$x_r(\zeta) := \exp(\zeta \cdot \mathrm{ad}\, e_r).$$

The matrices of $x_r(\zeta)$ w.r.t. $\mathcal{C}$ have entries in $\mathbb{Z}[\zeta]$.

This allows to define $x_r(t) \in \mathrm{Aut}(\mathfrak{g}_k)$ (by replacing ζ by $t \in k$).

$$G := \langle x_r(t) \mid r \in \Phi, t \in k \rangle \leq \mathrm{Aut}(\mathfrak{g}_k)$$

is the Chevalley group corresponding to $\mathfrak{g}$ over k.

Names such as $A_r(q)$, $B_r(q)$, $G_2(q)$, $E_6(q)$, etc. refer to the type of the root system Φ of $\mathfrak{g}$.

TWISTED GROUPS (TITS, STEINBERG, REE, 1957 – 61)

Chevalley's construction gives many of the finite groups of Lie type, but not all.

For example, $\mathrm{GU}_n(q)$ is not a Chevalley group in this sense.

However, $\mathrm{GU}_n(q)$ is obtained from the Chevalley group $\mathrm{GL}_n(q^2)$ by twisting:

Let σ denote the automorphism $(a_{ij}) \mapsto (a_{ij}^q)^{-tr}$ of $\mathrm{GL}_n(q^2)$. Then

$$\mathrm{GU}_n(q) = \mathrm{GL}_n(q^2)^\sigma := \{g \in \mathrm{GL}_n(q^2) \mid \sigma(g) = g\}.$$

Similar constructions give the twisted groups ${}^2E_6(q)$, ${}^3D_4(q)$, ${}^2B_2(2^{2m+1})$, ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$.

(${}^2B_2(2^{2m+1})$ was discovered in 1960 by Suzuki by a different method.)

LINEAR ALGEBRAIC GROUPS

Let $\bar{\mathbb{F}}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$.

A (linear) algebraic group **G** over $\bar{\mathbb{F}}_p$ is a closed subgroup of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$ for some n,

Closed: W.r.t. the Zariski topology, i.e. defined by polynomial equations.

EXAMPLES

(1) $\mathrm{SL}_n(\bar{\mathbb{F}}_p) = \{g \in \mathrm{GL}_n(\bar{\mathbb{F}}_p) \mid \det(g) = 1\}$.

(2) $\mathrm{SO}_n(\bar{\mathbb{F}}_p) = \{g \in \mathrm{SL}_n(\bar{\mathbb{F}}_p) \mid g^{tr} J g = J\} \quad (n = 2m + 1 \text{ odd})$.

G is semisimple, if it has no closed connected soluble normal subgroup $\neq 1$.

G is reductive, if it has no closed connected unipotent normal subgroup $\neq 1$.

Semisimple algebraic groups are reductive.

FROBENIUS MAPS

Let $\mathbf{G} \leq \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ be a connected reductive algebraic group.

A standard Frobenius map of $\mathbf{G}$ is a homomorphism

$$F := F_q : \mathbf{G} \to \mathbf{G}$$

of the form $F_q((a_{ij})) = (a_{ij}^q)$ for some power q of p.

(This implicitly assumes that $(a_{ij}^q) \in \mathbf{G}$ for all $(a_{ij}) \in \mathbf{G}$.)

EXAMPLES

$\mathrm{SL}_n(\bar{\mathbb{F}}_p)$ *and* $\mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)$ *admit standard Frobenius maps* F_q *for all powers* q *of* p.

A Frobenius map $F : \mathbf{G} \to \mathbf{G}$ is a homomorphism such that F^m is a standard Frobenius map for some $m \in \mathbb{N}$.

FINITE REDUCTIVE GROUPS

Let **G** be a connected reductive algebraic group over $\bar{\mathbb{F}}_p$ and let F be a Frobenius map of **G**.

Then $\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$ is a finite group.

The pair $(\mathbf{G}, F)$ or the finite group $G := \mathbf{G}^F$ is called finite reductive group or finite group of Lie type.

EXAMPLES

Let q be a power of p and let $F = F_q$ be the corresponding standard Frobenius map of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$, $(a_{ij}) \mapsto (a_{ij}^q)$.

Then $\mathrm{GL}_n(\bar{\mathbb{F}}_p)^F = \mathrm{GL}_n(q)$, $\mathrm{SL}_n(\bar{\mathbb{F}}_p)^F = \mathrm{SL}_n(q)$, $\mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)^F = \mathrm{SO}_{2m+1}(q)$.

All groups of Lie type, except the Suzuki and Ree groups are obtained in this way by a **standard** Frobenius map.

In the following, $(\mathbf{G}, F)$ denotes a finite reductive group over $\bar{\mathbb{F}}_p$.

The Lang-Steinberg theorem

Theorem (Lang-Steinberg, 1956/1968)

If $\mathbf{G}$ is connected, the map $\mathbf{G} \to \mathbf{G}$, $g \mapsto g^{-1}F(g)$ is surjective.

The assumption that $\mathbf{G}$ is connected is crucial here.

Example

Let $\mathbf{G} = \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, and $F : (q_{ij}) \mapsto (a_{ij}^q)$, q a power of p.

Then there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{G}$ such that

$$\begin{bmatrix} a^q & b^q \\ c^q & d^q \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

The Lang-Steinberg theorem is used to derive structural properties of $\mathbf{G}^F$.

MAXIMAL TORI AND THE WEYL GROUP

A torus of $\mathbf{G}$ is a closed subgroup isomorphic to $\bar{\mathbb{F}}_p^* \times \cdots \times \bar{\mathbb{F}}_p^*$.

A torus is maximal, if it is not contained in any larger torus of $\mathbf{G}$.

Crucial fact: Any two maximal tori of $\mathbf{G}$ are conjugate.

DEFINITION

The Weyl group W of $\mathbf{G}$ is defined by $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$.

EXAMPLE

Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ and $\mathbf{T}$ the group of diagonal matrices. Then:

1. *$\mathbf{T}$ is a maximal torus of $\mathbf{G}$,*
2. *$N_{\mathbf{G}}(\mathbf{T})$ is the group of monomial matrices,*
3. *$W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ can be identified with the group of permutation matrices, i.e. $W \cong S_n$.*

Maximal tori of finite reductive groups

A maximal torus of $(\mathbf{G}, F)$ is a finite reductive group $(\mathbf{T}, F)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$.

A maximal torus of $G = \mathbf{G}^F$ is a subgroup T of the form $T = \mathbf{T}^F$ for some maximal torus $(\mathbf{T}, F)$ of $(\mathbf{G}, F)$.

Example

A Singer cycle is a maximal torus of $\mathrm{GL}_n(q)$. (This is an irreducible cyclic subgroup of $\mathrm{GL}_n(q)$ of order $q^n - 1$.)

The maximal tori of $(\mathbf{G}, F)$ are classified (up to conjugation in G) by F-conjugacy classes of W.

These are the orbits under the action $v.w \mapsto vwF(v)^{-1}$, $v, w \in W$.

THE CLASSIFICATION OF MAXIMAL TORI

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$.

Let $w \in W$, and $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ with $w = \dot{w}\mathbf{T}$.

By the Lang-Steinberg theorem, there is $g \in \mathbf{G}$ such that $\dot{w} = g^{-1}F(g)$.

One checks that ${}^{g}\mathbf{T}$ is F-stable, and so $({}^{g}\mathbf{T}, F)$ is a maximal torus of $(\mathbf{G}, F)$.

The map $w \mapsto ({}^{g}\mathbf{T}, F)$ induces a bijection between the set of F-conjugacy classes of W and the set of G-conjugacy classes of maximal tori of $(\mathbf{G}, F)$.

We say that ${}^{g}\mathbf{T}$ is obtained from $\mathbf{T}$ by twisting with w.

THE MAXIMAL TORI OF $\mathrm{GL}_n(q)$

Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ and $F = F_q$ a standard Frobenius morphism.

Then F acts trivially on $W = S_n$, i.e. the maximal tori of $G = \mathrm{GL}_n(q)$ are parametrized by partitions of n.

If $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of n, we write T_λ for the corresponding maximal torus.

We have

$$|T_\lambda| = (q^{\lambda_1} - 1)(q^{\lambda_2} - 1) \cdots (q^{\lambda_l} - 1).$$

Each factor $q^{\lambda_i} - 1$ of $|T_\lambda|$ corresponds to a cyclic direct factor of T_λ of this order.

THE STRUCTURE OF THE MAXIMAL TORI

Let $\mathbf{T}'$ be an F-stable maximal torus of $\mathbf{G}$, obtained by twisting the reference torus $\mathbf{T}$ with $w = \dot{w}\mathbf{T} \in W$.

I.e. there is $g \in \mathbf{G}$ with $g^{-1}F(g) = \dot{w}$ and $\mathbf{T}' = {}^{g}\mathbf{T}$.
Then

$$T' = (\mathbf{T}')^F \cong \mathbf{T}^{wF} := \{t \in \mathbf{T} \mid t = \dot{w}F(t)\dot{w}^{-1}\}.$$

Indeed, for $t \in \mathbf{T}$ we have $gtg^{-1} = F(gtg^{-1})$ $[= F(g)F(t)F(g)^{-1}]$ if and only if $t \in \mathbf{T}^{wF}$.

EXAMPLE

Let $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$, *and* $\mathbf{T}$ *the group of diagonal matrices.*
Let $w = (1, 2, \ldots, n)$ *be an n-cycle. Then*

$$\mathbf{T}^{wF} = \{\mathrm{diag}[t, t^q, \ldots, t^{q^{n-1}}] \mid t \in \bar{\mathbb{F}}_p, t^{q^n-1} = 1\},$$

and so $\mathbf{T}^{wF}$ *is cyclic of order* $q^n - 1$.

BN-PAIRS

This axiom system was introduced by Jaques Tits to allow a uniform treatment of groups of Lie type.

DEFINITION

The subgroups B and N of the group G form a BN-pair, if:

1. $G = \langle B, N \rangle$;
2. $T := B \cap N$ *is normal in* N;
3. $W := N/T$ *is generated by a set* S *of involutions;*
4. *If* $\dot{s} \in N$ *maps to* $s \in S$ *(under* $N \to W$*), then* $\dot{s}B\dot{s} \neq B$;
5. *For each* $n \in N$ *and* $\dot{s}$ *as above,* $(B\dot{s}B)(BnB) \subseteq B\dot{s}nB \cup BnB$.

W is called the Weyl group of the *BN*-pair G. It is a Coxeter group with Coxeter generators S (more on this later).

THE *BN*-PAIR OF $\mathrm{GL}_n(k)$ AND OF $\mathrm{SO}_n(k)$

Let k be a field and $G = \mathrm{GL}_n(k)$. Then G has a BN-pair with:

- B: group of upper triangular matrices;
- N: group of monomial matrices;
- $T = B \cap N$: group of diagonal matrices;
- $W = N/T \cong S_n$: group of permutation matrices.

Let n be odd and let $\mathrm{SO}_n(k) = \{g \in \mathrm{SL}_n(k) \mid g^{tr} J g = J\}$ be the orthogonal group.

If B, N are as above, then

$$B \cap \mathrm{SO}_n(k), N \cap \mathrm{SO}_n(k)$$

is a BN-pair of $\mathrm{SO}_n(k)$.

SPLIT *BN*-PAIRS OF CHARACTERISTIC p

Let G be a group with a BN-pair (B, N).

This is said to be a split BN-pair of characteristic p, if the following additional hypotheses are satisfied:

6. $B = UT$ with $U = O_p(B)$, the largest normal p-subgroup of B, and T a complement of U.
7. $\bigcap_{n \in N} nBn^{-1} = T$. (Recall $T = B \cap N$.)

EXAMPLES

1. *A semisimple algebraic group over $\bar{\mathbb{F}}_p$ and a finite group of Lie type of characteristic p have split BN-pairs of characteristic p.*
2. *If $G = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ or $\mathrm{GL}_n(q)$, q a power of p, then U is the group of upper triangular unipotent matrices.*

   In the latter case, U is a Sylow p-subgroup of G.

Parabolic subgroups and Levi subgroups

Let G be a group with a split BN-pair of characteristic p.

Any conjugate of B is called a Borel subgroup of G.

A parabolic subgroup of G is one containing a Borel subgroup.

Let $P \leq G$ be a parabolic subgroup. Then

$$P = U_P L$$

with

- $U_P = O_p(P)$ is the largest normal p-subgroup of P.
- L is a complement to U_P in P.

This is called a Levi decomposition of P, and L is a Levi subgroup of G.

A Levi subgroup is itself a group with a split BN-pair of characteristic p.

EXAMPLES FOR PARABOLIC SUBGROUPS

In classical groups, parabolic subgroups are the stabilisers of isotropic subspaces.

Let $G = \mathrm{GL}_n(q)$, and $(\lambda_1, \ldots, \lambda_l)$ a partition of n. Then

$$P = \left\{ \begin{bmatrix} \mathrm{GL}_{\lambda_1}(q) & \star & \star \\ & \ddots & \star \\ & & \mathrm{GL}_{\lambda_l}(q) \end{bmatrix} \right\}$$

is a typical parabolic subgroup of G. A corresponding Levi subgroup is

$$L = \left\{ \begin{bmatrix} \mathrm{GL}_{\lambda_1}(q) & & \\ & \ddots & \\ & & \mathrm{GL}_{\lambda_l}(q) \end{bmatrix} \right\} \cong GL_{\lambda_1}(q) \times \cdots \times \mathrm{GL}_{\lambda_l}(q).$$

$B = UT$ with T the diagonal matrices and U the upper triangular unipotent matrices is a Levi decompositon of B.

THE BRUHAT DECOMPOSITION

Let G be a group with a BN-pair. Then

$$G = \dot{\bigcup_{w \in W}} BwB$$

(we write $Bw := B\dot{w}$ if $\dot{w} \in N$ maps to $w \in W$ under $N \to W$).
This is called the Bruhat decompositon of G. (The Bruhat decompositon for $\mathrm{GL}_n(k)$ follows from the Gaussian algorithm.)
Now suppose that the BN-pair is split, $B = UT = TU$.
Let $w \in W$. Then $\dot{w}T = T\dot{w}$ since $T \lhd N$, and so $BwB = BwU$.
Moreover, there is a subgroup $U_w \in U$ such that $BwU = BwU_w$, with "uniqueness of expression".
If G, furthermore, is finite, this implies

$$|G| = |B| \sum_{w \in W} |U_w|.$$

THE ORDERS OF THE FINITE GROUPS OF LIE TYPE

Let G be a finite group of Lie type of characteristic p. Then G has a split BN-pair of characteristic p. Thus

$$|G| = |B| \sum_{w \in W} |U_w|.$$

We have $|B| = |U||T|$ and $|U_w| = q^{\ell(w)}$. Here, q is a power of p. Also $\ell(w)$ is the length of $w \in W$, i.e. the shortest word in the Coxeter generators S of W expressing w.

By a theorem of Solomon (1966) and Steinberg (1968):

$$\sum_{w \in W} q^{\ell(w)} = \prod_{i=1}^{r} \frac{q^{d_i} - 1}{q - 1},$$

where $d_1, \ldots, d_r$ are the degrees of the basic polynomial invariants of W. This gives the formulae for $|G|$.

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Thank you for your attention!