FINITE GROUPS OF LIE TYPE AND THEIR REPRESENTATIONS – LECTURE IV

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CONTENTS

Harish-Chandra Classification: Recollection

Throughout this lecture, let k be an algebraically closed field of characteristic $\ell \geq 0$, and let G be a finite group.

Recall that Harish-Chandra theory yields the following classification for a finite group of Lie type of characteristic $p \neq \ell$.

Theorem (Harish-Chandra, Lusztig, Geck-H.-Malle)

$$\{ V \mid V \text{ simple } kG\text{-module} \} / \text{isomorphism}$$

$$\updownarrow$$

$$\left\{ (L, M, \theta) \;\middle|\; \begin{array}{c} L \text{ Levi subgroup of } G \\ M \text{ simple, cuspidal } kL\text{-module} \\ \theta \text{ irred. } k\text{-rep'n of } \mathcal{H}(L, M) \end{array} \right\} / \text{conjugacy}$$

PROBLEMS IN HARISH-CHANDRA THEORY

The above theorem leads to the three tasks:

1. Determine the cuspidal pairs (L, M).
2. For each of these, “compute” $\mathcal{H}(L, M)$.
3. Classify irreducible k-representations of $\mathcal{H}(L, M)$.

State of the art in case $\ell > 0$:

- $\mathcal{H}(L, M)$ is an Iwahori-Hecke algebra corresponding to an “extended” Coxeter group (Howlett-Lehrer (1980), Geck-H.-Malle (1996)), namely $W_G(L, M)$; parameters of $\mathcal{H}(L, M)$ not known in general.
- $G = \mathrm{GL}_n(q)$; everything known (Dipper-James, 1980s)
- G classical group, ℓ “linear”; everything known (Gruber-H., 1997).
- In general, classification of cuspidal pairs open.

EXAMPLE: $SO_{2m+1}(q)$ (GECK-H.-MALLE (1996))

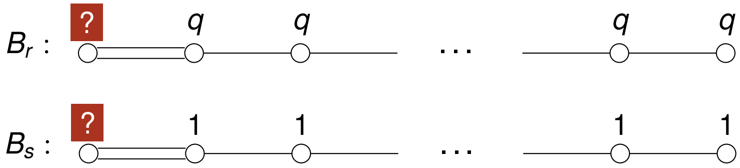
Let $G = SO_{2m+1}(q)$, assume that $\ell > m$, and put $e := \min\{i \mid \ell \text{ divides } q^i - 1\}$, the order of q in $\mathbb{F}_\ell^*$.

Any Levi subgroup L of G containing a cuspidal unipotent (see below) module M is of the form

$$L = SO_{2m'+1}(q) \times GL_1(q)^r \times GL_e(q)^s.$$

In this case $W_G(L, M) \cong W(B_r) \times W(B_s)$, where $W(B_j)$ denotes a Weyl group of type B_j.

Moreover, $\mathcal{H}(L, M) \cong \mathcal{H}_{k,\mathbf{q}}(B_r) \otimes \mathcal{H}_{k,\mathbf{q}}(B_s)$, with $\mathbf{q}$ as follows:



BRAUER CHARACTERS

From now on suppose that $\ell > 0$.

Let $\mathfrak{X}$ be a k-representation of G of degree d.

The character $\chi_{\mathfrak{X}}$ of $\mathfrak{X}$ as defined in Lecture 3 does not convey all the desired information, e.g.,

$\chi_{\mathfrak{X}}(1)$ only gives the degree d of $\mathfrak{X}$ modulo ℓ.

Instead one considers the Brauer character $\varphi_{\mathfrak{X}}$ of $\mathfrak{X}$.

This is obtained by consistently lifting the eigenvalues of the matrices $\mathfrak{X}(g)$ for $g \in G_{\ell'}$ to characteristic 0. (Here, $G_{\ell'}$ is the set of ℓ-regular elements of G.)

Thus $\varphi_{\mathfrak{X}} : G_{\ell'} \to K$, where K is a suitable field with $\mathrm{char}(K) = 0$, and $\varphi_{\mathfrak{X}}(g) =$ sum of the eigenvalues of $\mathfrak{X}(g)$ (viewed as elements of K).

In particular, $\varphi_{\mathfrak{X}}(1)$ equals the degree of $\mathfrak{X}$.

THE BRAUER CHARACTER TABLE

If $\mathfrak{X}$ is irreducible, $\varphi_{\mathfrak{X}}$ is called an irreducible Brauer character.

Put $\mathrm{IBr}_\ell(G) :=$ set of irreducible Brauer characters of G, $\mathrm{IBr}_\ell(G) = \{\varphi_1, \ldots, \varphi_n\}$.

(If $\ell \nmid |G|$, then $\mathrm{IBr}_\ell(G) = \mathrm{Irr}(G)$.)

Let $g_1, \ldots, g_n$ be representatives of the conjugacy classes contained in $G_{\ell'}$ (same n as above!).

The square matrix

$$[\varphi_i(g_j)]_{1 \leq i,j \leq l}$$

is called the Brauer character table or ℓ-modular character table of G.

THE 13-MODULAR CHARACTER TABLE OF $\mathrm{SL}_3(3)$

Let $G = \mathrm{SL}_3(3)$. Then $|G| = 5\,616 = 2^4 \cdot 3^3 \cdot 13$.

EXAMPLE (THE 13-MODULAR CHARACTER TABLE OF $\mathrm{SL}_3(3)$)

	$1a$	$2a$	$3a$	$3b$	$4a$	$6a$	$8a$	$8b$
φ_1	1	1	1	1	1	1	1	1
φ_2	11	3	2	-1	-1	0	-1	-1
φ_3	13	-3	4	1	1	0	-1	-1
φ_4	16	0	-2	1	0	0	0	0
φ_5	26	2	-1	-1	2	-1	0	0
φ_6	26	-2	-1	-1	0	1	$\sqrt{-2}$	$-\sqrt{-2}$
φ_7	26	-2	-1	-1	0	1	$-\sqrt{-2}$	$\sqrt{-2}$
φ_8	39	-1	3	0	-1	-1	1	1

Goals and Results

Aim

Describe all Brauer character tables of all finite simple groups and related finite groups.

In contrast to the case of ordinary character tables (cf. Lecture 3), this is wide open:

1. For alternating groups: complete up to A_{17}
2. For groups of Lie type: only partial results
3. For sporadic groups up to McL and other "small" groups (of order $\leq 10^9$): *An Atlas of Brauer Characters*, Jansen, Lux, Parker, Wilson, 1995

   More information is available on the web site of the Modular Atlas Project:
   (http://www.math.rwth-aachen.de/~MOC/)

THE DECOMPOSITION NUMBERS

For $\chi \in \mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_m\}$, write $\hat{\chi}$ for the restriction of χ to $G_{\ell'}$.

Then there are integers $d_{ij} \geq 0$, $1 \leq i \leq m$, $1 \leq j \leq n$ such that $\hat{\chi}_i = \sum_{j=1}^{l} d_{ij}\, \varphi_j$.

These integers are called the decomposition numbers of G modulo ℓ.

The matrix $D = [d_{ij}]$ is the decomposition matrix of G.

PROPERTIES OF BRAUER CHARACTERS

Two irreducible k-representations are equivalent if and only if their Brauer characters are equal.

$\mathrm{IBr}_\ell(G)$ is linearly independent (in $\mathrm{Maps}(G_{\ell'}, K)$) and so the decomposition numbers are uniquely determined.

The elementary divisors of D are all 1 (i.e., the decomposition map defined by $\chi \mapsto \hat{\chi}$ is surjective). Thus:

Knowing $\mathrm{Irr}(G)$ and D is equivalent to knowing $\mathrm{Irr}(G)$ and $\mathrm{IBr}_\ell(G)$.

If G is ℓ-soluble, $\mathrm{Irr}(G)$ and $\mathrm{IBr}_\ell(G)$ can be sorted such that D has shape

$$D = \left[\begin{array}{c} I_n \\ \hline D' \end{array}\right],$$

where I_n is the $(n \times n)$ identity matrix (Fong-Swan theorem).

THE 13-DECOMPOSITION NUMBERS OF $SL_3(3)$

		φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
		1	11	13	16	26	26	26	39
χ_1	1	1	.	.	.	.	.	.	.
χ_2	12	1	1	.	.	.	.	.	.
χ_3	13	.	.	1	.	.	.	.	.
χ_4	16	.	.	.	1	.	.	.	.
χ_5	16	.	.	.	1	.	.	.	.
χ_6	16	.	.	.	1	.	.	.	.
χ_7	16	.	.	.	1	.	.	.	.
χ_8	26	.	.	.	.	1	.	.	.
χ_9	26	.	.	.	.	.	1	.	.
χ_{10}	26	.	.	.	.	.	.	1	.
χ_{11}	27	.	1	.	1	.	.	.	.
χ_{12}	39	.	.	.	.	.	.	.	1

UNIPOTENT BRAUER CHARACTERS

The concept of decomposition numbers can be used to define unipotent Brauer characters of a finite reductive group.

Let $(\mathbf{G}, F)$ be a finite reductive group of characteristic p.
(Recall that $\mathrm{char}(k) = \ell \neq p$.)

Recall that

$$\mathrm{Irr}^u(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi \text{ occurs in } R^{\mathbf{G}}_{\mathbf{T},\mathbf{1}} \text{ for some max. torus } \mathbf{T} \text{ of } \mathbf{G}\}.$$

This yields a definition of $\mathrm{IBr}^u_\ell(G)$.

DEFINITION (UNIPOTENT BRAUER CHARACTERS)

$$\mathrm{IBr}^u_\ell(G) = \{\varphi_j \in \mathrm{IBr}_\ell(G) \mid d_{ij} \neq 0 \textit{ for some } \chi_i \in \mathrm{Irr}^u(G)\}.$$

The elements of $\mathrm{IBr}^u_\ell(G)$ *are called the unipotent Brauer characters of* G.

A simple kG-module is unipotent, if its Brauer character is.

Jordan decomposition of Brauer characters

The investigations are guided by the following main conjecture.

Conjecture

Suppose that $Z(\mathbf{G})$ is connected. Then there is a labelling

$$\mathrm{IBr}_\ell(G) \leftrightarrow \{\varphi_{s,\mu} \mid s \in G^* \text{ semisimple}, \ell \nmid |s|, \mu \in \mathrm{IBr}_\ell^u(C_{G^*}(s))\},$$

such that $\varphi_{s,\mu}(1) = |G^* : C_{G^*}(s)|_{p'}\, \mu(1)$.

*Moreover, D can be computed from the decomposition numbers of **unipotent** characters of the various $C_{G^*}(s)$.*

Known to be true for $\mathrm{GL}_n(q)$ (Dipper-James, 1980s) and in many other cases (Bonnafé-Rouquier, 2003).

The truth of this conjecture would reduce the computation of decomposition numbers to unipotent characters.

Consequently, we will restrict to this case in the following.

The unipotent decomposition matrix

Put $D^u :=$ restriction of D to $\mathrm{Irr}^u(G) \times \mathrm{IBr}^u_\ell(G)$.

Theorem (Geck-H., 1991; Geck, 1993)

(Some conditions apply.)

$|\mathrm{Irr}^u(G)| = |\mathrm{IBr}^u_\ell(G)|$ *and* D^u *is invertible over* $\mathbb{Z}$.

Conjecture (Geck, 1997)

(Some conditions apply.) With respect to suitable orderings of $\mathrm{Irr}^u(G)$ *and* $\mathrm{IBr}^u_\ell(G)$, D^u *has shape*

$$\begin{bmatrix} 1 & & & \\ \star & 1 & & \\ \vdots & \vdots & \ddots & \\ \star & \star & \star & 1 \end{bmatrix}.$$

This would give a canonical bijection $\mathrm{Irr}^u(G) \longleftrightarrow \mathrm{IBr}^u_\ell(G)$.

About Geck's Conjecture

Geck's conjecture on D^u is known to hold for

- $\mathrm{GL}_n(q)$ (Dipper-James, 1980s)
- $\mathrm{GU}_n(q)$ (Geck, 1991)
- G a classical group and ℓ "linear" (Gruber-H., 1997)
- $\mathrm{Sp}_4(q)$ (White, 1988 – 1995)
- $\mathrm{Sp}_6(q)$ (An-H., 2006)
- $G_2(q)$ (Shamash-H., 1989 – 1992)
- $F_4(q)$ (Köhler, 2006)
- $E_6(q)$ (Geck-H., 1997; Miyachi, 2008)
- Steinberg triality groups ${}^3D_4(q)$ (Geck, 1991)
- Suzuki groups (for general reasons)
- Ree groups (Himstedt-Huang, 2009)

LINEAR PRIMES, I

Let $(\mathbf{G}, F)$ be a finite reductive group, where $F = F_q$ is the standard Frobenius morphism $(a_{ij}) \mapsto (a_{ij}^q)$.

Put $e := \min\{i \mid \ell \text{ divides } q^i - 1\}$, the order of q in $\mathbb{F}_\ell^*$.

If G is classical ($\neq \mathrm{GL}_n(q)$) and e is odd, ℓ is linear for G.

EXAMPLE

$G = \mathrm{SO}_{2m+1}(q)$, $|G| = q^{m^2}(q^2-1)(q^4-1)\cdots(q^{2m}-1)$.
If $\ell \,||G|$ and $\ell \nmid q$, then $\ell \mid q^{2d}-1$ for some minimal d.
Thus $\ell \mid q^d - 1$ (ℓ linear and $e = d$) or $\ell \mid q^d + 1$ ($e = 2d$).

Now $\mathrm{Irr}^u(G)$ is a union of Harish-Chandra series $\mathcal{E}_1, \ldots, \mathcal{E}_r$.

THEOREM (FONG-SRINIVASAN, 1982, 1989)

Suppose that $G \neq \mathrm{GL}_n(q)$ is classical and that ℓ is linear. Then $D^u = \mathrm{diag}[\Delta_1, \ldots, \Delta_r]$ with square matrices Δ_i corresponding to $\mathcal{E}_i$.

LINEAR PRIMES, II

Let $\Delta := \Delta_i$ be one of the decomposition matrices from above.

Then the rows and columns of Δ are labelled by bipartitions of a for some integer a. (Harish-Chandra theory.)

THEOREM (GRUBER-H., 1997)

In general,

$$\Delta = \begin{bmatrix} \Lambda_0 \otimes \Lambda_a & & & & \\ & \ddots & & & \\ & & \Lambda_i \otimes \Lambda_{a-i} & & \\ & & & \ddots & \\ & & & & \Lambda_a \otimes \Lambda_0 \end{bmatrix}$$

Here $\Lambda_i \otimes \Lambda_{a-i}$ is the Kronecker product of matrices, and Λ_i is the ℓ-modular unipotent decomposition matrix of $\mathrm{GL}_i(q)$.

The v-Schur algebra

Let v be an indeterminate an put $A := \mathbb{Z}[v, v^{-1}]$.

Dipper and James (1989) have defined a remarkable A-algebra $\mathcal{S}_{A,v}(S_n)$, called the generic v-Schur algebra, such that:

1. $\mathcal{S}_{A,v}(S_n)$ is free and of finite rank over A.
2. $\mathcal{S}_{A,v}(S_n)$ is constructed from the generic Iwahori-Hecke algebra $\mathcal{H}_{A,v}(S_n)$, which is contained in $\mathcal{S}_{A,v}(S_n)$ as a subalgebra (with a different unit).
3. $\mathbb{Q}(v) \otimes_A \mathcal{S}_{A,v}(S_n)$ is a quotient of the quantum group $\mathcal{U}_v(\mathfrak{gl}_n)$.

THE q-SCHUR ALGEBRA

Let $G = \mathrm{GL}_n(q)$.

Then $D^u = (d_{\lambda,\mu})$, with $\lambda, \mu \in \mathcal{P}_n$.

Let $\mathcal{S}_{A,v}(S_n)$ be the v-Schur algebra, and let $\mathcal{S} := \mathcal{S}_{k,q}(S_n)$ be the finite-dimensional k-algebra obtained by specializing v to the image of $q \in k$.

This is called the q-Schur algebra, and satisfies:

1. $\mathcal{S}$ has a set of (finite-dimensional) standard modules $\mathbf{S}^\lambda$, indexed by $\mathcal{P}_n$.
2. The simple $\mathcal{S}$-modules $\mathbf{D}^\lambda$ are also labelled by $\mathcal{P}_n$.
3. If $[\mathbf{S}^\lambda : \mathbf{D}^\mu]$ denotes the multiplicity of $\mathbf{D}^\mu$ as a composition factor in $\mathbf{S}^\lambda$, then $[\mathbf{S}^\lambda : \mathbf{D}^\mu] = d_{\lambda,\mu}$.

As a consequence, the $d_{\lambda,\mu}$ are bounded independently of q and of ℓ.

CONNECTIONS TO DEFINING CHARACTERISTICS, I

Let $\mathcal{S}_{k,q}(S_n)$ be the q-Schur algebra introduced above.

Suppose that $\ell \mid q-1$ so that $q \equiv 1 (\mathrm{mod}\ \ell)$.

Then $\mathcal{S}_{k,q}(S_n) \cong \mathcal{S}_k(S_n)$, where $\mathcal{S}_k(S_n)$ is the Schur algebra studied by J. A. Green (1980).

A partition λ of n may be viewed as a dominant weight of $\mathrm{GL}_n(k)$ [$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \leftrightarrow \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_m\varepsilon_m$].

Thus there are corresponding $k\mathrm{GL}_n(k)$-modules $V(\lambda)$ and $L(\lambda)$.

If λ and μ are partitions of n, we have

$$[V(\lambda) : L(\mu)] = [\mathbf{S}^\lambda : \mathbf{D}^\mu] = d_{\lambda,\mu}.$$

The first equality comes from the significance of the Schur algebra, the second from that of the q-Schur algebra.

Connections to defining characteristics, II

Thus the ℓ-modular decomposition numbers of $\mathrm{GL}_n(q)$ for prime powers q with $\ell \mid q-1$, determine the composition multiplicities of **certain** simple modules $L(\mu)$ in **certain** Weyl modules $V(\lambda)$ of $\mathrm{GL}_n(k)$, namely if λ and μ are partitions of n.

Facts (Schur, Green)

Let λ and μ be partitions with at most n parts.

1. $[V(\lambda) : L(\mu)] = 0$, *if λ and μ are partitions of different numbers.*
2. *If λ and μ are partitions of $r \geq n$, then the composition multiplicity $[V(\lambda) : L(\mu)]$ is the same in $\mathrm{GL}_n(k)$ and $\mathrm{GL}_r(k)$.*

Hence the ℓ-modular decomposition numbers of **all** $\mathrm{GL}_r(q)$, $r \geq 1$, $\ell \mid q-1$ determine the composition multiplicities of **all** Weyl modules of $\mathrm{GL}_n(k)$.

CONNECTIONS TO SYMMETRIC GROUP REPR'S

As for the Schur algebra, there are standard kS_n- modules S^λ, called Specht modules, labelled by the partitions λ of n.

The simple kS_n-modules D^μ are labelled by the ℓ-regular partitions μ of n (no part of μ is repeated ℓ or more times).

The ℓ-modular decomposition numbers of S_n are the $[S^\lambda : D^\mu]$.

THEOREM (JAMES, 1980)

$[S^\lambda : D^\mu] = [V(\lambda) : L(\mu)]$, if μ is ℓ-regular (notation from $\mathrm{GL}_n(k)$ case).

THEOREM (ERDMANN, 1995)

For partitions λ, μ of n, there are ℓ-regular partition $t(\lambda)$, $t(\mu)$ of $\ell n + (\ell - 1)n(n-1)/2$ such that

$$[V(\lambda) : L(\mu)] = [S^{t(\lambda)} : D^{t(\mu)}].$$

AMAZING CONCLUSION

Recall that ℓ is a fixed prime and k an algebraically closed field of characteristic ℓ.

Each of the following three families of numbers can be determined from any one of the others:

1. The ℓ-modular decomposition numbers of S_n for all n.
2. The ℓ-modular decomposition numbers of the unipotent characters of $\mathrm{GL}_n(q)$ for all primes powers q with $\ell \mid q-1$ and all n.
3. The composition multiplicities of the simple $k\mathrm{GL}_n(k)$-modules in the Weyl modules of $\mathrm{GL}_n(k)$ for all n.

Thus all these problems are really hard.

JAMES' CONJECTURE

Let $G = \mathrm{GL}_n(q)$. Recall that $e = \min\{i \mid \ell \text{ divides } q^i - 1\}$. James has computed all matrices D^u for $n \leq 10$.

CONJECTURE (JAMES, 1990)

If $e\ell > n$, then D^u only depends on e (neither on ℓ nor q).

THEOREM

(1) The conjecture is true for $n \leq 10$ (James, 1990).

(2) If $\ell >> 0$, D^u only depends on e (Geck, 1992).

In fact, Geck proved $D^u = D_e D_\ell$ for two square matrices D_e and D_ℓ, and that $D_\ell = I$ for $\ell >> 0$.

THEOREM (LASCOUX-LECLERC-THIBON; ARIKI; VARAGNOLO-VASSEROT (1996 – 99))

The matrix D_e can be computed from the canonical basis of a certain highest weight module of the quantum group $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$.

A unipotent decomposition matrix for $\mathrm{GL}_5(q)$

Let $G = \mathrm{GL}_5(q)$, $e = 2$ (i.e., $\ell \mid q+1$ but $\ell \nmid q-1$), and assume $\ell > 2$. Then D^u equals

(5)	1						
$(4,1)$		1					
$(3,2)$			1				
$(3,1^2)$	1		1	1			
$(2^2,1)$			1	1	1		
$(2,1^3)$		1				1	
(1^5)	1			1			1

The triangular shape defines φ_λ, $\lambda \in \mathcal{P}_5$.

ON THE DEGREE POLYNOMIALS

The degrees of the φ_λ are "polynomials in q".

λ	$\varphi_\lambda(1)$
(5)	1
$(4,1)$	$q(q+1)(q^2+1)$
$(3,2)$	$q^2(q^4+q^3+q^2+q+1)$
$(3,1^2)$	$(q^2+1)(q^5-1)$
$(2^2,1)$	$(q^3-1)(q^5-1)$
$(2,1^3)$	$q(q+1)(q^2+1)(q^5-1)$
(1^5)	$q^2(q^3-1)(q^5-1)$

THEOREM (BRUNDAN-DIPPER-KLESHCHEV, 2001)

The degrees of $\chi_\lambda(1)$ and of $\varphi_\lambda(1)$ as polynomials in q are the same.

GENERICITY

Let $\{G(q) \mid q$ a prime power with $\ell \nmid q\}$ be a series of finite groups of Lie type, e.g. $\{\mathrm{GU}_n(q)\}$ or $\{\mathrm{SO}_{2m+1}(q)\}$ (n, respectively m fixed).

QUESTION

Is an analogue of James' conjecture true for $\{G(q)\}$?

If **yes**, only finitely many matrices D^u to compute (finitely many e's and finitely many "small" ℓ's).

The following is a weaker form.

CONJECTURE

The entries of D^u are bounded independently of q and ℓ.

This conjecture is known to be true for $\mathrm{GL}_n(q)$ (Dipper-James), G classical and ℓ linear (Gruber-H., 1997), $\mathrm{GU}_3(q)$, $\mathrm{Sp}_4(q)$ (Okuyama-Waki, 1998, 2002).

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