IMPRIMITIVE IRREDUCIBLE REPRESENTATIONS OF FINITE SIMPLE GROUPS

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Contents

The project

This is a joint project with William J. Husen and Kay Magaard.

Project

Classify the pairs $(G, G \to \mathrm{SL}(V))$ such that

1. *G is a finite quasisimple group,*
2. *V a finite dimensional vector space over some field K,*
3. *$G \to \mathrm{SL}(V)$ is absolutely irreducible and imprimitive.*

Explanations

1. *G is quasisimple, if $G = G'$ and $G/Z(G)$ is simple.*
2. *$G \to \mathrm{SL}(V)$ is imprimitive, if $V = V_1 \oplus \cdots \oplus V_m$, $m > 1$, and the action of G permutes the V_i transitively.*

   We call $H := \mathrm{Stab}_G(V_1)$ a block stabilizer.

   We have $V \cong \mathrm{Ind}_H^G(V_1) := KG \otimes_{KH} V_1$ as KG-modules.

Primitivity and tensor products

Theorem (Aschbacher, 2000)

Let K be an algebraically closed field, let G_i be finite groups, and let V_i be finite-dimensional KG_i-modules for $i = 1, 2$.

Then the $K[G_1 \times G_2]$-module $V_1 \otimes_K V_2$ is primitive, if and only if V_i is a primitive KG_i-module for $i = 1, 2$.

The proof is trickier than one would expect.

Example (I forgot, who told me this)

Let $G = J_2$ and $K = \mathbb{C}$ (and we replace modules by characters).

$\chi := \chi_2 = 14$ and $\psi := \chi_{18} = 225$ are primitive, but

$$\chi \cdot \psi = \mathrm{Ind}_H^G(6)$$

is imprimitive, where $H = 2^{2+4} : (3 \times S_3)$.

MOTIVATION I: MAXIMAL SUBGROUPS

Let K be a finite field and V a f.d. K-vector space.
Let $X \leq \mathrm{SL}(V)$ be a classical group, e.g., $X = \mathrm{Sp}(V), \mathrm{SO}(V)$.
Let $G \leq X$ be finite, quasisimple, such that

1. $\varphi : G \to X \leq \mathrm{SL}(V)$ is absolutely irreducible, and
2. not realizable over a smaller field.

[$\varphi : G \to \mathrm{SL}(V)$ is realizable over a smaller field, if φ factors as

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & \mathrm{SL}(V) \\ & \searrow^{\varphi_0} & \uparrow \nu \\ & & \mathrm{SL}(V_0) \end{array}$$

for some proper subfield $K_0 \leq K$, a K_0-vector space V_0 with $V = K \otimes_{K_0} V_0$, and a representation $\varphi_0 : G \to \mathrm{SL}(V_0)$.]

Is $N_X(G)$ a maximal subgroup of X?

Some obstructions

The following obstructions (for the maximality of $N_X(G)$), and many more, arise from Aschbacher's subgroup classification (1984) [cf. Eamonn O'Brien's plenary talk].

$\mathcal{C}_2$-obstruction: $\varphi : N_X(G) \to X \leq \mathrm{SL}(V)$ is **imprimitive**.

Then $N_X(G) \lneq \mathrm{Stab}_X(\{V_1, \ldots, V_m\}) \lneq X$.

$\mathcal{C}_4$-obstruction: $\varphi : N_X(G) \to X \leq \mathrm{SL}(V)$ is **tensor decomposable**,

i.e., $V = U \otimes_K W$ and φ is equivalent to $\varphi_U \otimes \varphi_W$.

Then $N_X(G) \lneq X \cap (\mathrm{SL}(U) \otimes_K \mathrm{SL}(W)) \lneq X$.

$\mathcal{S}$-obstruction: There is a quasisimple group H such that $N_X(G) \lneq H \lneq X$. (Thus $\mathrm{Res}_G^H(V)$ is absolutely irreducible.)

An example: The Mathieu group M_{11}

Let X be a finite classical group.

Let $\varphi : M_{11} \to X$ be absolutely irreducible, faithful, and not realizable over a smaller field. (All such (φ, X) are known.)

Put $G := \varphi(M_{11})$. Then $N_X(G) = Z(X) \times G$.

Is $Z(X) \times G$ maximal in X?

NO, except for $\varphi : M_{11} \to \mathrm{SL}_5(3)$.

Examples

(1) $M_{11} \to A_{11} \to \mathrm{SO}_{10}^{+}(3)'$ *($\mathcal{S}$-obstruction).*

(2) $M_{11} \to \mathrm{SO}_{55}(\ell)$ *is imprimitive,* $\ell \geq 5$ *($\mathcal{C}_2$-obstruction).*

(3) Also: $M_{11} \to M_{12} \to A_{12} \to \mathrm{SO}_{11}(\ell) \to \mathrm{SO}_{55}(\ell)$, $\ell \geq 5$.

(4) $M_{11} \to 2.M_{12} \to \mathrm{SL}_{10}(3)$ *($\mathcal{S}$-obstruction).*

(5) $M_{11} \to \mathrm{SL}_5(3) \to \mathrm{SO}_{24}^{-}(3)'$ *($\mathcal{S}$-obstruction).*

What about $\varphi : M \to \mathrm{SO}_{196882}(2)$? ($M$: Monster)

Motivation II: Matrix groups computation

The following algorithmic problem arises in the "matrix groups computation" project [cf. Eamonn O'Brien's plenary talk].

Let K be a finite field, $x_1, \ldots, x_r \in \mathrm{GL}_n(K)$, $G := \langle x_1, \ldots, x_r \rangle$.

Through preliminary computations one knows

1. G acts absolutely irreducibly on $V = K^n$,
2. G is "nearly" simple,
3. the isomorphism type of the non-abelian simple composition factor of G.

Decide whether G acts primitively on V.

A table look-up in our lists might help to answer this question.

SPORADIC SIMPLE GROUPS

Complete list of examples for sporadic simple groups:

G	$\dim(V)$	$N_G(V_1)$	V_1	$\mathrm{char}(K)$
M_{11}	11 55	$A_6.2_3$ $3^2 : Q_8.2$	1_2 1_3	$\neq 2, 3$
M_{12}	66 120	$A_6.2^2$ M_{11}	1_3 $10_2, 10_3$	$\neq 2, 3$ $\neq 2, 3, 11$
M_{22}	231	$2^4 : A_6$	$3_1, 3_2$	3
M_{24}	1 771	$2^6 : 3.S_6$	1_2	$\neq 2, 3$
McL	9 625	$U_4(3)$	$35_1, 35_2$	$\neq 2, 3$
Co_2	1 288 000 2 095 875	$U_6(2) : 2$ $2^{10} : M_{22} : 2$	$560_1, 560_2$ $45_2, 45_4$	$\neq 2, 3, 11$ $\neq 2, 7, 11$

There are a few more examples for covering groups of these.

THE ALTERNATING GROUPS; $K = \mathbb{C}$

We replace modules by characters, $\mathrm{Irr}(G)$ denotes the set of irreducible $\mathbb{C}$-characters of G.

THEOREM (DRAGOMIR DJOKOVIĆ, JERRY MALZAN, 1976)

Suppose that $G = A_n$, $n \geq 10$, and let $\chi \in \mathrm{Irr}(G)$ be imprimitive. Then one of the following holds.

1. $n = m^2 + 1$ *and* $\chi = \mathrm{Res}_G^{S_n}(\zeta^\lambda)$ *with* $\lambda = (m+1, m^{m-1})$.

   Also, $\chi = \mathrm{Ind}_{A_{n-1}}^G(\chi_1)$ *with* χ_1 *a constituent of* $\mathrm{Res}_{A_{n-1}}^{S_{n-1}}(\zeta^\mu)$ *with* $\mu = (m^m)$.
2. $n = 2m$ *and* $\chi = \mathrm{Res}_G^{S_n}(\zeta^\lambda)$ *with* $\lambda = (m+1, 1^{m-1})$.

   Also, $\chi = \mathrm{Ind}_{N_G(S_m \times S_m)}^G(\chi_1)$ *with* $\chi_1(1) = 1$.

The classification for A_n is complete in all characteristics.

The covering groups of the alternating groups; $K = \mathbb{C}$

Theorem (Daniel Nett, Felix Noeske, 2009)

Suppose that $G = 2.A_n$, $n \geq 10$, is the covering group of A_n, and let $\psi \in \mathrm{Irr}(G)$ be imprimitive.

Then $n = 1 + m(m+1)/2$, and $\psi = \mathrm{Res}_G^{2.S_n}(\sigma^\lambda)$ with $\lambda = (m+1, m-1, m-2, \ldots, 1)$.

Also, $\psi = \mathrm{Ind}_{2.A_{n-1}}^G(\psi_1)$ with ψ_1 a constituent of $\mathrm{Res}_{2.A_{n-1}}^{2.S_{n-1}}(\sigma^\mu)$ with $\mu = (m, m-1, \ldots, 1)$.

THE COVERING GROUPS OF THE ALTERNATING GROUPS; $\mathrm{char}(K) > 0$

Let K be an algebraically closed field of characteristic $\neq 0$.

THEOREM (DANIEL NETT, FELIX NOESKE, 2009)

Suppose that $G = 2.A_n$, $n \geq 10$, is the covering group of A_n.
Let $H \leq G$ be a maximal subgroup such that $\mathrm{Ind}_H^G(V_1)$ is irreducible, for some KH-module V_1.
Then $H/Z(G) \leq A_n$ either is an ***intransitive*** *subgroup of A_n, or $n = 2m$ is even and $H/Z(G) = (S_m \wr S_2) \cap A_n$.*

The classification for $2.A_n$ in these cases is still open.

FINITE REDUCTIVE GROUPS

Let $\mathbf{G}$ denote a reductive algebraic group over $\mathbf{F}$, an algebraically closed field, $\mathrm{char}(\mathbf{F}) = p > 0$.

Let F denote a Frobenius morphism of $\mathbf{G}$ with respect to some $\mathbb{F}_q$-structure of $\mathbf{G}$.

Then $G := \mathbf{G}^F$ is a finite reductive group of characteristic p.

An F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is split, if $\mathbf{L}$ is a Levi complement in an F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$.

Such a pair $(\mathbf{L}, \mathbf{P})$ gives rise to a parabolic subgroup $P = \mathbf{P}^F$ of G with Levi complement $L = \mathbf{L}^F$.

REDUCTIVE GROUPS IN DEFINING CHARACTERISTICS

The following result of Seitz contains the classification in defining characteristic.

THEOREM (GARY SEITZ, 1988)

Let G be a finite reductive, quasisimple group of characteristic p.

Suppose that V is an irreducible, imprimitive $\mathbf{F}G$-module.

Then G is one of

$$\mathrm{SL}_2(5), \mathrm{SL}_2(7), \mathrm{SL}_3(2), \mathrm{Sp}_4(3),$$

and V is the Steinberg module.

The main reduction theorem

Let G be a finite reductive group of characteristic p.

Suppose that G

1. is quasisimple,
2. does not have an exceptional Schur multiplier,
3. is not isomorphic to a finite reductive group of a different characteristic.

Let K be an algebraically closed field with $\mathrm{char}(K) \neq p$.

Theorem (Husen-H.-Magaard, 2013)

Let G and K be as above. Let $H \leq G$ be a maximal subgroup. Suppose that $\mathrm{Ind}_H^G(V_1)$ is irreducible for some KH-module V_1.

Then $H = P$ is a parabolic subgroup of G.

Some easy characteristic-free criteria

Let G be a finite group, $H \leq G$, and K a field.
Let V_1 be a KH-module such that $V := \mathrm{Ind}_H^G(V_1)$ is irreducible.
Then

1. $[G : H]$ divides $\dim(V)$.
2. $|H|^2 \geq |G|$.
3. For all $t \in G \setminus H$, the group ${}^tH \cap H$ is **not** centralized by t. In particular ${}^tH \cap H \neq \{1\}$ for all $t \in G$.
4. Suppose that $H = C_G(a)$ for some $a \in G$. Then $t \notin \langle {}^ta, a \rangle$ for all $t \in G \setminus H$.

Proof of 1: Clear, since $\dim(V) = [G : H]\dim(V_1)$.
Proof of 2: $[G : H]^2 \leq \dim(V)^2 \leq |G|$.
Proof of 3: This is a consequence of Mackey's theorem.
Proof of 4: For $t \in G$, ${}^tH \cap H = C_G({}^ta, a)$. Hence $t \notin \langle {}^ta, a \rangle$ for $t \in G \setminus H$, since such a t does not centralize ${}^tH \cap H$ by 3.

NON-PARABOLIC BLOCK STABILIZERS

Large subgroups of finite reductive groups are **in general** parabolic subgroups.

There are, however, many exceptions, causing a lot of trouble.

EXAMPLE

Let $G = \mathrm{Sp}_{2m}(q)$ *with* m *even and* $q > 3$ *odd, and let*

$H = \langle H_0, s\rangle$ *with* $H_0 = \mathrm{Sp}_m(q) \times \mathrm{Sp}_m(q)$ *and* $s = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$.

Then $H_0 = C_G(a)$ *with* $a = \begin{bmatrix} \alpha I_m & 0 \\ 0 & \alpha^{-1} I_m \end{bmatrix}$, *where* $\langle\alpha\rangle = \mathbb{F}_q^*$.

Put $t := \begin{bmatrix} I_m & N \\ N & I_m \end{bmatrix}$ *with* $N := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Then $t \in \langle {}^t a, a\rangle$, *hence* t *centralizes* ${}^t H_0 \cap H_0$.

Finally, $t \in C_G(s) \setminus H$ *and* ${}^t H_0 \cap s H_0 = \emptyset$, *thus* $t \in C_G({}^t H \cap H)$.

Parabolic block stabilizers

Let G be a finite reductive, quasisimple group of characteristic p, and let K be an algebraically closed field with $\mathrm{char}(K) \neq p$.

According to our main reduction theorem, we may restrict our investigation to parabolic subgroups.

Proposition (Husen-H.-Magaard, 2013)

Let P be a parabolic subgroup of G with unipotent radical U.

Let V_1 be a KP-module such that $\mathrm{Ind}_P^G(V_1)$ is irreducible. Then U is in the kernel of V_1.

In other words, $\mathrm{Ind}_P^G(V_1)$ is Harish-Chandra induced.

This allows to apply Harish-Chandra theory to our classification problem, reducing certain aspects to Weyl groups.

SKETCH PROOF OF PROPOSITION

PROPOSITION

Let P be a parabolic subgroup of G with unipotent radical U. Let V_1 be a KP-module such that $\mathrm{Ind}_P^G(V_1)$ is irreducible. Then U is in the kernel of V_1.

Proof: (Sketch) Let L be a Levi complement of U in P.

Chose a head composition factor V_2 of $\mathrm{Res}_L^P(V_1)$.

Let Q be the opposite parabolic subgroup of P, so $P \cap Q = L$.

Mackey's theorem yields a non-trivial homomorphism $\mathrm{Ind}_P^G(V_1) \to \mathrm{Ind}_Q^G(\tilde{V}_2)$, where $\tilde{V}_2 = \mathrm{Infl}_L^Q(V_2)$.

As $\mathrm{Ind}_P^G(V_1)$ is simple, and $\dim(\mathrm{Ind}_Q^G(\tilde{V}_2)) \leq \dim(\mathrm{Ind}_P^G(V_1))$, this implies that

$$\mathrm{Ind}_P^G(V_1) \cong \mathrm{Ind}_Q^G(\tilde{V}_2).$$

It follows that $\dim(V_1) = \dim(V_2)$.

A consequence for maximal subgroups

Let X be a finite classical group on the vector space V.

Let $G \leq X$ be a quasisimple reductive group such that

1. $\varphi : G \to X \leq \mathrm{SL}(V)$ is absolutely irreducible,
2. $V = \mathrm{Ind}_P^G(V_1)$ for some parabolic subgroup P of G,
3. the G-conjugacy class of P is invariant under $N_X(G)$.

Then $N_X(G)$ is **not** a maximal subgroup of X.

Indeed, putting $H := N_X(G)$, we get $H = GN_H(P)$ by 3.

We have $V = V_1 \oplus \cdots \oplus V_m$, the V_i being permuted by G.

By the proposition, $V_1 = C_V(U)$, where U is the unipotent radical of P.

Now $N_H(P)$ stabilizes U, hence fixes V_1.

Thus $H = GN_H(P)$ permutes the V_i.

Harish-Chandra induction and imprimitivity

Let G be a finite reductive, quasisimple group of characteristic p, and let K be an algebraically closed field with $\mathrm{char}(K) \neq p$.

By Harish-Chandra theory, a large proportion of irreducible KG-modules are imprimitive.

Remark

Let L be a Levi complement of the parabolic subgroup P of G, and let V_1 be an irreducible KL-module which is rigid. This means, roughly, that the stabilizer of V_1 in $N_G(L)$ equals L. Then $\mathrm{Ind}_P^G(\mathrm{Infl}_L^P(V_1))$ is irreducible.

Example

$G = \mathrm{GL}_n(q)$, $L = \mathrm{GL}_m(q) \times \mathrm{GL}_{n-m}(q)$ with $m \neq n - m$. Then every irreducible KL-module is rigid.

Asymptotics

Assume from now on that $K = \mathbb{C}$ (our results are best in this case).

Let $G_m(q) = \mathrm{SL}_m(q)$ or $G_m(q) = \mathrm{Sp}_{2m}(q)$. Put

$$f(m,q) := \frac{|\mathrm{Irr}_i(G_m(q))|}{|\mathrm{Irr}(G_m(q))|},$$

where $\mathrm{Irr}_i(G_m(q)) = \{\chi \in \mathrm{Irr}(G_m(q)) \mid \chi \text{ is imprimitive}\}$.

Then $f(m) := \lim_{q\to\infty} f(m,q)$ exists an we have:

1. $f(m) = 1 - 1/m$ if $G_m(q) = \mathrm{SL}_m(q)$,
2. $f(m) = 1 - \frac{1\cdot3\cdot5\cdots(2m-1)}{2^m m!}$, if $G_m(q) = \mathrm{Sp}_{2m}(q)$ [Lübeck].

In each case, $\lim_{m\to\infty} f(m) = 1$.

Analogous results hold for the other classical groups.

EXAMPLE: $SL_2(q)$, q EVEN

	C_1	C_2	$C_3(a)$	$C_4(b)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(m)$	$q+1$	1	$\zeta^{am}+\zeta^{-am}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi^{bn}-\xi^{-bn}$

$$a, m = 1, \ldots, (q-2)/2, \qquad b, n = 1, \ldots, q/2,$$

The characters $\chi_3(m)$ are imprimitive, the others are primitive.

Number of irreducible characters: $q+1$.

Number of imprimitive irreducible characters: $q/2-1$.

LUSZTIG SERIES

Let $G = \mathbf{G}^F$ be a finite reductive group.
Let $G^* = \mathbf{G}^{*F}$ denote a dual reductive group.
We have

$$\mathrm{Irr}(G) = \bigcup_{[s]} \mathcal{E}(G, [s]),$$

a disjoint union into rational Lusztig series ($[s]$ runs through the G^*-conjugacy classes of semisimple elements of G^*).

THEOREM (HUSEN-H.-MAGAARD, 2013)

If $C_{\mathbf{G}^*}(s)$ is contained in a proper split Levi subgroup of $\mathbf{G}^*$, every element of $\mathcal{E}(G, [s])$ is Harish-Chandra induced.

*Suppose that $C_{\mathbf{G}^*}(s)$ is connected and **not** contained in a proper split Levi subgroup of $\mathbf{G}^*$.*

Then every element of $\mathcal{E}(G, [s])$ is Harish-Chandra primitive.

In particular, the elements of $\mathcal{E}(G, [1])$ are HC-primitive.

THE CLASSIFICATION FOR $\mathrm{GL}_n(q)$

Let $G = \mathrm{GL}_n(q)$. Then $\mathbf{G} = \mathbf{G}^*$.

Let $s \in G^* = G$ be semisimple. Then $C_{\mathbf{G}^*}(s)$ is connected.

THEOREM (HUSEN-H.-MAGAARD, 2013)

If the minimal polynomial of s is irreducible, then every element of $\mathcal{E}(G, [s])$ is Harish-Chandra primitive.

Otherwise, every element of $\mathcal{E}(G, [s])$ is Harish-Chandra induced.

Notice that the minimal polynomial of s is irreducible if and only if $C_G(s) \cong \mathrm{GL}_m(q^d)$ for integers m, d with $md = n$.

EXAMPLE FOR THE DESCENT FROM $\mathrm{GL}_n(q)$ TO $\mathrm{SL}_n(q)$

The descent from $\mathrm{GL}_n(q)$ to $\mathrm{SL}_n(q)$ is not so easy to describe.

EXAMPLE (CÉDRIC BONNAFÉ)

Suppose that q is odd, let $G = \mathrm{GL}_4(q)$ and P a parabolic subgroup with Levi complement $L = \mathrm{GL}_2(q) \times \mathrm{GL}_2(q)$.

Let $\mathbf{1}$ denote the trivial character and $\mathbf{1}^-$ the unique linear character of $\mathrm{GL}_2(q)$ of order 2.

Then $\chi := \mathrm{Ind}_P^G(\mathrm{Infl}_L^P(\mathbf{1} \otimes \mathbf{1}^-))$ is irreducible, hence imprimitive.

*However, $\mathrm{Res}^G_{\mathrm{SL}_4(q)}(\chi) = \psi_1 + \psi_2$, with irreducible, **primitive** characters ψ_1, ψ_2.*

THEOREM (HUSEN, H., MAGAARD, 2013)

Let $\chi \in \mathrm{Irr}(\mathrm{GL}_n(q))$ be Harish-Chandra primitive.

Then $\mathrm{Res}^{\mathrm{GL}_n(q)}_{\mathrm{SL}_n(q)}(\chi)$ is irreducible and Harish-Chandra primitive.

DESCENT FROM $\mathrm{GL}_n(q)$ TO $\mathrm{SL}_n(q)$

Let $G = \mathrm{SL}_n(q)$, $s \in G^* = \mathrm{PGL}_n(q)$ semisimple.

There is a bijection

$$\mathrm{Irr}(W(s)^F) \to \mathcal{E}(G, [s]), \quad \eta \mapsto \chi_\eta,$$

where $W(s)$ is the "Weyl group" of $C_{\mathbf{G}^*}(s)$ (Bonnafé).

Suppose that $\mathcal{E}(G, [s])$ contains Harish-Chandra primitive **and** imprimitive characters.

Then $W(s)^F = S \colon \langle \gamma \rangle$, with $S = S_m \times \cdots \times S_m$, and γ permuting the e factors S_m of S transitively, and $em \mid n$.

THEOREM (H.-MAGAARD)

$\chi_\eta \in \mathcal{E}(G, [s])$ is primitive, if and only if $\mathrm{Res}_S^{S \colon \langle \gamma \rangle}(\eta)$ is irreducible.

Thank you for listening!