REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE

LECTURE II: DELIGNE-LUSZTIG THEORY AND SOME APPLICATIONS

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Three Cases

Aim

Classify all irreducible representations of all finite simple groups and related finite groups.

In the following, let $G = G^F$ be a finite reductive group of characteristic $p$ and let $k$ be an algebraically closed field. It is natural to distinguish three cases:

1. $\text{char}(k) = p$ (usually $k = \overline{F}_p$); defining characteristic (cf. Jantzen’s lectures)

2. $\text{char}(k) = 0$; ordinary representations

3. $\text{char}(k) > 0$, $\text{char}(k) \neq p$; non-defining characteristic

Today I will talk about Case 2, so assume that $\text{char}(k) = 0$ from now on.
A SIMPLIFICATION: CHARACTERS

Let $V, V'$ be $kG$-modules.

The character afforded by $V$ is the map

$$\chi_V : G \to k, \quad g \mapsto \text{Trace}(g|V).$$

Characters are class functions.

$V$ and $V'$ are isomorphic, if and only if $\chi_V = \chi_{V'}$.

$Irr(G) := \{\chi_V | V \text{ simple } kG\text{-module}\}$: irreducible characters

$C$: set of representatives of the conjugacy classes of $G$

The square matrix

$$[\chi(g)]_{\chi \in \text{Irr}(G), g \in C}$$

is the ordinary character table of $G$. 
An example: The alternating group $A_5$

### Example (The Character Table of $A_5 \cong \text{SL}_2(4)$)

<table>
<thead>
<tr>
<th></th>
<th>$1a$</th>
<th>$2a$</th>
<th>$3a$</th>
<th>$5a$</th>
<th>$5b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>$A$</td>
<td>$*A$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>$*A$</td>
<td>$A$</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$$A = (1 - \sqrt{5})/2, \quad *A = (1 + \sqrt{5})/2$$

$1 \in 1a, \quad (1, 2)(3, 4) \in 2a, \quad (1, 2, 3) \in 3a,$

$(1, 2, 3, 4, 5) \in 5a, \quad (1, 3, 5, 2, 4) \in 5b$
GOALS AND RESULTS

AIM

Describe all ordinary character tables of all finite simple groups and related finite groups.

Almost done:

1. For alternating groups: Frobenius, Schur
2. For groups of Lie type: Green, Deligne, Lusztig, Shoji, . . . (only “a few” character values missing)
3. For sporadic groups and other “small” groups: Atlas of Finite Groups, Conway, Curtis, Norton, Parker, Wilson, 1986
**The generic character table for SL$_2(q)$, $q$ even**

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3(a)$</th>
<th>$C_4(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>$q$</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\chi_3(m)$</td>
<td>$q + 1$</td>
<td>1</td>
<td>$\zeta^{am} + \zeta^{-am}$</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4(n)$</td>
<td>$q - 1$</td>
<td>$-1$</td>
<td>0</td>
<td>$-\xi^{bn} - \xi^{-bn}$</td>
</tr>
</tbody>
</table>

$a, m = 1, \ldots, (q - 2)/2, \quad b, n = 1, \ldots, q/2,$

$\zeta := \exp\left(\frac{2\pi \sqrt{-1}}{q-1}\right), \quad \xi := \exp\left(\frac{2\pi \sqrt{-1}}{q+1}\right)$

$$\begin{bmatrix} \mu^a & 0 \\ 0 & \mu^{-a} \end{bmatrix} \in C_3(a) \quad (\mu \in \mathbb{F}_q \text{ a primitive } (q - 1)\text{th root of } 1)$$

$$\begin{bmatrix} \nu^b & 0 \\ 0 & \nu^{-b} \end{bmatrix} \in C_4(b) \quad (\nu \in \mathbb{F}_{q^2} \text{ a primitive } (q + 1)\text{th root of } 1)$$

Specialising $q$ to 4, gives the character table of SL$_2(4) \cong A_5$. 
Drinfeld’s example

The cuspidal simple $k\text{SL}_2(q)$-modules have dimensions $q - 1$ and $(q - 1)/2$ (the latter only occur if $p$ is odd).

How to construct these?

Consider the affine curve

$$C = \{(x, y) \in \bar{\mathbb{F}}_p^2 \mid xy^q - x^q y = 1\}.$$

$G = \text{SL}_2(q)$ acts on $C$ by linear change of coordinates. Hence $G$ also acts on the étale cohomology group

$$H^1_c(C, \bar{\mathbb{Q}}_\ell),$$

where $\ell$ is a prime different from $p$.

It turns out that the simple $\bar{\mathbb{Q}}_\ell G$-submodules of $H^1_c(C, \bar{\mathbb{Q}}_\ell)$ are the cuspidal ones (here $k = \bar{\mathbb{Q}}_\ell$).
Let $\ell$ be a prime different from $p$ and put $k := \overline{\mathbb{Q}}_\ell$.

Recall that $G = G^F$ is a finite reductive group.

Deligne and Lusztig (1976) construct for each pair $(T, \theta)$, where $T$ is an $F$-stable maximal torus of $G$, and $\theta \in \text{Irr}(T^F)$, a generalised character $R^G_T(\theta)$ of $G$.

(A generalised character of $G$ is an element of $\mathbb{Z}[\text{Irr}(G)]$.

Let $(T, \theta)$ be a pair as above.

Choose a Borel subgroup $B = TU$ of $G$ with Levi subgroup $T$.

(In general $B$ is not $F$-stable.)

Consider the Deligne-Lusztig variety associated to $U$,

$$Y_U = \{g \in G \mid g^{-1}F(g) \in U\}.$$ 

This is an algebraic variety over $\overline{\mathbb{F}}_p$. 

The finite groups $G = G^F$ and $T = T^F$ act on $Y_U$, and these actions commute.

Thus the étale cohomology group $H^i_c(Y_U, \mathbb{Q}_\ell)$ is a $\mathbb{Q}_\ell G$-module-$\mathbb{Q}_\ell T$,

and so its $\theta$-isotypic component $H^i_c(Y_U, \mathbb{Q}_\ell)_\theta$ is a $\mathbb{Q}_\ell G$-module, whose character is denoted by $\operatorname{ch} H^i_c(Y_U, \mathbb{Q}_\ell)_\theta$.

Only finitely many of the vector spaces $H^i_c(Y_U, \mathbb{Q}_\ell)$ are $\neq 0$.

Now put

$$R^G_T(\theta) = \sum_i (-1)^i \operatorname{ch} H^i_c(Y_U, \mathbb{Q}_\ell)_\theta.$$ 

This is a Deligne-Lusztig generalised character.
Properties of Deligne-Lusztig characters

The above construction and the following facts are due to Deligne and Lusztig (1976).

 Facts

Let \((\mathbf{T}, \theta)\) be a pair as above. Then

1. \(R^G_T(\theta)\) is independent of the choice of \(B\) containing \(T\).

2. If \(\theta\) is in general position, i.e. \(N_G(T, \theta)/T = \{1\}\), then \(\pm R^G_T(\theta)\) is an irreducible character.

 Facts (Continued)

3. For \(\chi \in \text{Irr}(G)\), there is a pair \((\mathbf{T}, \theta)\) such that \(\chi\) occurs in \(R^G_T(\theta)\).
A generalisation

Instead of a torus $T$ one can consider any $F$-stable Levi subgroup $L$ of $G$.

**Warning:** $L$ does in general not give rise to a Levi subgroup of $G$ as used in my first lecture.

Consider a parabolic subgroup $P$ of $G$ with Levi complement $L$ and unipotent radical $U$, not necessarily $F$-stable.

The corresponding Deligne-Lusztig variety $Y_U$ is defined as before: $Y_U = \{ g \in G \mid g^{-1}F(g) \in U \}$.

This is related to the one defined by Jean Michel: $Y_U \mapsto \{ gU \in G/U \mid gU \cap F(gU) \neq \emptyset \}$, $g \mapsto gU$.

One gets a **Lusztig-induction** map $R_{L \subset P}^G : \mathbb{Z}[\text{Irr}(L)] \to \mathbb{Z}[\text{Irr}(G)]$, $\mu \mapsto R_{L \subset P}^G(\mu)$. 
The above construction and the following facts are due to Lusztig (1976).

Let $L$ be an $F$-stable Levi subgroup of $G$ contained in the parabolic subgroup $P$, and let $\mu \in \mathbb{Z}[\text{Irr}(L)]$.

**Facts**

1. $R^G_{L \subset P}(\mu)(1) = \pm [G : L]_p \cdot \mu(1)$.

2. If $P$ is $F$-stable, then $R^G_{L \subset P}(\mu) = R^G_L(\mu)$ is the Harish-Chandra induced character.

3. Jean Michel's version of $Y_U$ yields the same map $R^G_{L \subset P}$.

It is not known, whether $R^G_{L \subset P}$ is independent of $P$, but it is conjectured that this is so.
**UNIPOTENT CHARACTERS**

**Definition (Lusztig)**

A character \( \chi \) of \( G \) is called **unipotent**, if \( \chi \) is irreducible, and if \( \chi \) occurs in \( R_T^G(1) \) for some \( F \)-stable maximal torus \( T \) of \( G \), where \( 1 \) denotes the trivial character of \( T = T^F \).

We write \( \text{Irr}^u(G) \) for the set of unipotent characters of \( G \).

The above definition of unipotent characters uses étale cohomology groups.

So far, no elementary description known, except for \( \text{GL}_n(q) \); see below.

Lusztig classified \( \text{Irr}^u(G) \) in all cases, **independently** of \( q \).

Harish-Chandra induction preserves unipotent characters (i.e. \( \text{Irr}^u(G) \) is a union of Harish-Chandra series), so it suffices to construct the **cuspidal** unipotent characters.
The unipotent characters of $GL_n(q)$

Let $G = GL_n(q)$ and $T$ the torus of diagonal matrices. Then $Irr^u(G) = \{ \chi \in Irr(G) \mid \chi \text{ occurs in } R_T^G(1) \}$. Moreover, there is bijection

$$\mathcal{P}_n \leftrightarrow Irr^u(G), \quad \lambda \leftrightarrow \chi_\lambda,$$

where $\mathcal{P}_n$ denotes the set of partitions of $n$. This bijection arises from $\text{End}_{kG}(R_T^G(1)) \cong \mathcal{H}_{k,q}(S_n) \cong kS_n$.

The degrees of the unipotent characters are “polynomials in $q$”:

$$\chi_\lambda(1) = q^{d(\lambda)} \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{\prod_{h(\lambda)}(q^h - 1)},$$

with a certain $d(\lambda) \in \mathbb{N}$, and where $h(\lambda)$ runs through the hook lengths of $\lambda$. 
## Degrees of the Unipotent Characters of $\text{GL}_5(q)$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\chi_{\lambda}(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5)</td>
<td>1</td>
</tr>
<tr>
<td>(4, 1)</td>
<td>$q(q+1)(q^2+1)$</td>
</tr>
<tr>
<td>(3, 2)</td>
<td>$q^2(q^4 + q^3 + q^2 + q + 1)$</td>
</tr>
<tr>
<td>(3, 1$^2$)</td>
<td>$q^3(q^2 + 1)(q^2 + q + 1)$</td>
</tr>
<tr>
<td>(2$^2$, 1)</td>
<td>$q^4(q^4 + q^3 + q^2 + q + 1)$</td>
</tr>
<tr>
<td>(2, 1$^3$)</td>
<td>$q^6(q+1)(q^2+1)$</td>
</tr>
<tr>
<td>(1$^5$)</td>
<td>$q^{10}$</td>
</tr>
</tbody>
</table>
JORDAN DECOMPOSITION OF CONJUGACY CLASSES

This is a model classification for \( \text{Irr}(G) \).

For \( g \in G \) with Jordan decomposition \( g = us = su \), we write \( C^G_{u,s} \) for the \( G \)-conjugacy class containing \( g \).

This gives a labelling

\[
\{\text{conjugacy classes of } G\} \leftrightarrow \{ C^G_{s,u} \mid s \text{ semisimple, } u \in C_G(s) \text{ unipotent}\}.
\]

(In the above, the labels \( s \) and \( u \) have to be taken modulo conjugacy in \( G \) and \( C_G(s) \), respectively.)

Moreover, \( |C^G_{s,u}| = |G : C_G(s)||C^{(s)}_{1,u}| \).

This is the Jordan decomposition of conjugacy classes.
**Example: The general linear group once more**

$G = \text{GL}_n(q), \ s \in G$ semisimple. Then

$$C_G(s) \cong \text{GL}_{n_1}(q^{d_1}) \times \text{GL}_{n_2}(q^{d_2}) \times \cdots \times \text{GL}_{n_m}(q^{d_m})$$

with $\sum_{i=1}^m n_i d_i = n$. (This gives finitely many class types.)

Thus it suffices to classify the set of unipotent conjugacy classes $\mathcal{U}$ of $G$.

By Linear Algebra we have

$$\mathcal{U} \longleftrightarrow \mathcal{P}_n = \{\text{partitions of } n\}$$

$$C^G_{1,u} \longleftrightarrow (\text{sizes of Jordan blocks of } u)$$

This classification is generic, i.e., independent of $q$.

In general, i.e. for other groups, it depends slightly on $q$. 
Let $G^*$ denote the reductive group dual to $G$. If $G$ is determined by the root datum $(X, \Phi, X^\vee, \Phi^\vee)$, then $G^*$ is defined by the root datum $(X^\vee, \Phi^\vee, X, \Phi)$.

**Examples**

1. If $G = \text{GL}_n(\overline{F}_p)$, then $G^* = G$.
2. If $G = \text{SO}_{2m+1}(\overline{F}_p)$, then $G^* = \text{Sp}_{2m}(\overline{F}_p)$.

$F$ gives rise to a Frobenius map on $G^*$, also denoted by $F$.

**Main Theorem (Lusztig; Jordan dec. of char’s, 1984)**

Suppose that $Z(G)$ is connected. Then there is a bijection

$$\text{Irr}(G) \longleftrightarrow \{\chi_{s,\lambda} \mid s \in G^* \text{ semisimple}, \lambda \in \text{Irr}^u(C_{G^*}(s))\}$$

(where the $s \in G^*$ are taken modulo conjugacy in $G^*$).

Moreover, $\chi_{s,\lambda}(1) = |G^*: C_{G^*}(s)|_{p'} \lambda(1)$. 
Suppose that \( s \in G^* \) is semisimple such that \( L^* := C_{G^*}(s) \) is a Levi subgroup of \( G^* \).

This is the generic situation, e.g. it is always the case if \( G = GL_n(\overline{F}_p) \) or if \(|s|\) is divisible by good primes only.

Then there is an \( F \)-stable Levi subgroup \( L \) of \( G \), dual to \( L^* \).

By Lusztig’s classification of unipotent characters, \( \text{Irr}^u(L) \) and \( \text{Irr}^u(L^*) \) can be identified.

Moreover, there is a linear character \( \hat{s} \in \text{Irr}(L) \), “dual” to \( s \in Z(L^*) \), such that

\[
\chi_{s,\lambda} = \pm R_{L \subset P}^G(\hat{s}\lambda)
\]

for all \( \lambda \in \text{Irr}^u(L) \leftrightarrow \text{Irr}^u(L^*) \) (and some choice of \( P \)).
The irreducible characters of $GL_n(q)$

Let $G = GL_n(q)$. Then

$$\text{Irr}(G) = \{ \chi_{s, \lambda} \mid s \in G \text{ semisimple}, \lambda \in \text{Irr}^u(C_G(s)) \}.$$ 

We have $C_G(s) \cong GL_{n_1}(q^{d_1}) \times GL_{n_2}(q^{d_2}) \times \cdots \times GL_{n_m}(q^{d_m})$ with $\sum_{i=1}^m n_i d_i = n$.

Thus $\lambda = \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_m$ with $\lambda_i \in \text{Irr}^u(GL_{n_i}(q^{d_i})) \leftrightarrow \mathcal{P}_{n_i}$.

Moreover,

$$\chi_{s, \lambda}(1) = \frac{(q^n - 1) \cdots (q - 1)}{\prod_{i=1}^m [(q^{d_i n_i} - 1) \cdots (q^{d_i} - 1)]} \prod_{i=1}^m \lambda_i(1).$$
### Degrees of the irreducible characters of $\text{GL}_3(q)$

<table>
<thead>
<tr>
<th>$C_G(s)$</th>
<th>$\lambda$</th>
<th>$\chi_{s,\lambda}(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{GL}_1(q^3)$</td>
<td>(1)</td>
<td>$(q - 1)^2(q + 1)$</td>
</tr>
<tr>
<td>$\text{GL}_1(q^2) \times \text{GL}_1(q)$</td>
<td>(1) $\boxtimes$ (1)</td>
<td>$(q - 1)(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\text{GL}_1(q)^3$</td>
<td>(1) $\boxtimes$ (1) $\boxtimes$ (1)</td>
<td>$(q + 1)(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\text{GL}_2(q) \times \text{GL}_1(q)$</td>
<td>(2) $\boxtimes$ (1)</td>
<td>$q^2 + q + 1$</td>
</tr>
<tr>
<td></td>
<td>(1, 1) $\boxtimes$ (1)</td>
<td>$q(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\text{GL}_3(q)$</td>
<td>(3)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(2, 1)</td>
<td>$q(q + 1)$</td>
</tr>
<tr>
<td></td>
<td>(1, 1, 1)</td>
<td>$q^3$</td>
</tr>
</tbody>
</table>

(This example was already known to Steinberg.)
Lusztig series

Lusztig (1988) also obtained a Jordan decomposition for \( \text{Irr}(G) \) in case \( Z(G) \) is not connected, e.g. if \( G = \text{SL}_n(\overline{\mathbb{F}}_p) \) or \( G = \text{Sp}_{2m}(\overline{\mathbb{F}}_p) \) with \( p \) odd.

For such groups, \( C_{G^*}(s) \) is not always connected, and the problem is to define \( \text{Irr}^u(C_{G^*}(s)) \), the unipotent characters.

The Jordan decomposition yields a partition

\[
\text{Irr}(G) = \bigcup_{(s) \subset G^*} \mathcal{E}(G, s),
\]

where \((s)\) runs through the semisimple \( G^*\)-conjugacy classes of \( G^* \) and \( s \in (s) \).

By definition, \( \mathcal{E}(G, s) = \{ \chi_{s, \lambda} \mid \lambda \in \text{Irr}^u(C_{G^*}(s)) \} \).

For example \( \mathcal{E}(G, 1) = \text{Irr}^u(G) \).

The sets \( \mathcal{E}(G, s) \) are called rational Lusztig series.
1. The Jordan decompositions of conjugacy classes and characters allow for the construction of generic character tables in all cases.

2. Let \( \{ G(q) \mid q \text{ a prime power} \} \) be a series of finite groups of Lie type, e.g. \( \{ \text{GU}_n(q) \} \) or \( \{ \text{SL}_n(q) \} \) (\( n \) fixed, \( q \) variable). Then there exists a finite set \( \mathcal{D} \) of polynomials in \( \mathbb{Q}[x] \) s.t.: If \( \chi \in \text{Irr}(G(q)) \), then there is \( f \in \mathcal{D} \) with \( \chi(1) = f(q) \).
Let $G$ be a finite group and let $\mathcal{O}$ be a complete dvr of residue characteristic $\ell > 0$.

Then

$$\mathcal{O} G = B_1 \oplus \cdots \oplus B_r,$$

with indecomposable 2-sided ideals $B_i$, the blocks of $\mathcal{O} G$ (or $\ell$-blocks of $G$).

Write

$$1 = e_1 + \cdots + e_r$$

with $e_i \in B_i$. Then the $e_i$ are exactly the primitive idempotents in $Z(\mathcal{O} G)$ and $B_i = \mathcal{O} Ge_i = e_i \mathcal{O} G = e_i \mathcal{O} Ge_i$.

$\chi \in \text{Irr}(G)$ belongs to $B_i$, if $\chi(e_i) \neq 0$.

This yields a partition of $\text{Irr}(G)$ into $\ell$-blocks.
A result of Fong and Srinivasan

Let $G = \text{GL}_n(q)$ or $U_n(q)$, where $q$ is a power of $p$.

As for $\text{GL}_n(q)$, the unipotent characters of $U_n(q)$ are labelled by partitions of $n$.

Let $\ell \neq p$ be a prime and put

$$e := \begin{cases} 
\min \{ i \mid \ell \text{ divides } q^i - 1 \}, & \text{if } G = \text{GL}_n(q) \\
\min \{ i \mid \ell \text{ divides } (-q)^i - 1 \}, & \text{if } G = U_n(q). 
\end{cases}$$

(Thus $e$ is the order of $q$, respectively $-q$ in $\mathbb{F}_\ell^*$.)

Theorem (Fong-Srinivasan, 1982)

Two unipotent characters $\chi_\lambda, \chi_\mu$ of $G$ are in the same $\ell$-block of $G$, if and only if $\lambda$ and $\mu$ have the same $e$-core.

Fong and Srinivasan found a similar combinatorial description for the $\ell$-blocks of the other classical groups.
Let again $G$ be a finite reductive group of characteristic $p$ and let $\ell$ be a prime, $\ell \neq p$.

For a semisimple $\ell'$-element $s \in G^*$, define

$$E_\ell(G, s) := \bigcup_{t \in C_{G^*}(s)_\ell} \mathcal{E}(G, st).$$

**Theorem (Broué-Michel, 1989)**

$E_\ell(G, s)$ is a union of $\ell$-blocks of $G$. 
Let $G$ and $\ell$ be as above.

Suppose $G = G^F$ with $F(a_{ij}) = (a_{ij}^q)$ for some power $q$ of $p$.

Write $d$ for the order of $q$ in $F^*$. 

A $d$-cuspidal pair is a pair $(L, \zeta)$, where $L$ is an $F$-stable $d$-split Levi subgroup of $G$, and $\zeta \in \text{Irr}(L)$ is $d$-cuspidal.

**Theorem (Cabanes-Enguehard, 1994)**

(Some mild conditions apply.) Suppose that $B$ is an $\ell$-block of $G$ contained in $\mathcal{E}_\ell(G, 1)$, the union of unipotent blocks. Then there is a $d$-cuspidal pair $(L, \zeta)$ such that

$$B \cap \mathcal{E}(G, 1) = \{ \chi \in \text{Irr}^u(G) \mid \chi \text{ is a constituent of } R^G_{L \subset P}(\zeta) \}.$$ 

A similar description applies for $B \cap \mathcal{E}(G, t)$ with $t \in (G^*)_{\ell}$. 
Thank you for your listening!