REPRESENTATION THEORY FOR GROUPS OF LIE TYPE

LECTURE I: FINITE REDUCTIVE GROUPS

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Logic and Algorithms in Group Theory
September 10, 2018

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THE CLASSIFICATION OF THE FINITE SIMPLE GROUPS

THEOREM

Every finite simple group is

1. *one of* 26 *sporadic simple groups; or*
2. *a cyclic group of prime order; or*
3. *an alternating group* A_n *with* $n \geq 5$*; or*
4. *a finite group of Lie type.*

THE FINITE CLASSICAL GROUPS

Examples for finite groups of Lie type are the finite classical groups.
These are linear groups over finite fields, preserving a form of degree 1 or 2 (possibly trivial).

EXAMPLES

- $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{Sp}_{2m}(q)$, $\mathrm{SO}_{2m+1}(q)$ ... (q a prime power)
- E.g., $\mathrm{SO}_{2m+1}(q) = \{g \in SL_{2m+1}(q) \mid g^{tr}Jg = J\}$, with $J = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{F}_q^{2m+1\times 2m+1}$.
- Related groups, e.g., $\mathrm{SL}_n(q)$, $\mathrm{PSL}_n(q)$, $\mathrm{CSp}_{2m}(q)$ etc. are also classical groups.

Not all classical groups are simple, but closely related to simple goups, e.g. $\mathrm{SL}_n(q) \to \mathrm{PSL}_n(q) = \mathrm{SL}_n(q)/Z(SL_n(q))$.

EXCEPTIONAL GROUPS

There are groups of Lie type which are not classical, namely,

Exceptional groups: $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$
(q a prime power, the order of a finite field),

Twisted groups: ${}^2E_6(q)$, ${}^3D_4(q)$ (q a prime power),

Suzuki groups: ${}^2B_2(2^{2m+1})$ ($m \geq 0$),

Ree groups: ${}^2G_2(3^{2m+1})$, ${}^2F_4(2^{2m+1})$ ($m \geq 0$).

The names of these goups, e.g. $G_2(q)$ or $E_8(q)$ refer to simple complex Lie algebras or rather their root systems.

How are groups of Lie type constructed? What are their properties, important subgroups, orders, etc?

THE ORDERS OF SOME FINITE GROUPS OF LIE TYPE

$|\mathrm{GL}_n(q)| = q^{n(n-1)/2}(q-1)(q^2-1)(q^3-1)\cdots(q^n-1).$

$|\mathrm{GU}_n(q)| = q^{n(n-1)/2}(q+1)(q^2-1)(q^3+1)\cdots(q^n-(-1)^n).$

$|\mathrm{SO}_{2m+1}(q)| = q^{m^2}(q^2-1)(q^4-1)\cdots(q^{2m}-1).$

$|F_4(q)| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1).$

$|{}^2F_4(q)| = q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1) \quad (q = 2^{2m+1}).$

There is a systematic way to derive these order formulae.

Structure of the formulae:

$$|G| = q^N \prod_{i=1}^{m} \Phi_i(q)^{a_i},$$

where Φ_i is the i th cyclotomic polynomial and $a_i \in \mathbb{N}$.

FINITE GROUPS OF LIE TYPE VS. FINITE REDUCTIVE GROUPS

Finite reductive groups are groups of fixed points of a Frobenius morphism, acting on a reductive algebraic group (see below).

A finite reductive group is a finite group of Lie type.

$\mathrm{PSL}_n(q)$ is a finite group of Lie type, but **not** a finite reductive group.

In the following, we shall focus on finite reductive groups.

LINEAR ALGEBRAIC GROUPS

Let **F** denote the algebraic closure of the finite field $\mathbb{F}_p$.

A (linear) algebraic group **G** over **F** is a closed subgroup of $\mathrm{GL}_n(\mathbf{F})$ for some n.

Closed: W.r.t. the Zariski topology, i.e. defined by polynomial equations.

EXAMPLES

(1) $\mathrm{SL}_n(\mathbf{F}) = \{g \in \mathrm{GL}_n(\mathbf{F}) \mid \det(g) = 1\}$.

(2) $\mathrm{SO}_{2m+1}(\mathbf{F}) = \{g \in \mathrm{SL}_{2m+1}(\mathbf{F}) \mid g^{tr} J g = J\}$.

G is semisimple, if it has no closed connected soluble normal subgroup $\neq 1$.

G is reductive, if it has no closed connected unipotent normal subgroup $\neq 1$.

Semisimple algebraic groups are reductive.

FROBENIUS MAPS

Let $\mathbf{G} \leq \mathrm{GL}_n(\mathbf{F})$ be a connected reductive algebraic group.

A standard Frobenius map of $\mathbf{G}$ is a homomorphism

$$F := F_q : \mathbf{G} \to \mathbf{G}$$

of the form $F_q((a_{ij})) = (a_{ij}^q)$ for some power q of p.

(This implicitly assumes that $(a_{ij}^q) \in \mathbf{G}$ for all $(a_{ij}) \in \mathbf{G}$.)

EXAMPLES

$\mathrm{SL}_n(\mathbf{F})$ and $\mathrm{SO}_{2m+1}(\mathbf{F})$ admit standard Frobenius maps F_q for all powers q of p.

A Frobenius map $F : \mathbf{G} \to \mathbf{G}$ is a homomorphism such that F^m is a standard Frobenius map for some $m \in \mathbb{N}$.

FINITE REDUCTIVE GROUPS

Let $\mathbf{G}$ be a connected reductive algebraic group over $\mathbf{F}$ and let F be a Frobenius map of $\mathbf{G}$.

Then $\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$ is a finite group.

The pair $(\mathbf{G}, F)$ or the finite group $G := \mathbf{G}^F$ is called finite reductive group (of characteristic p).

EXAMPLES

Let q be a power of p and let $F = F_q$ be the corresponding standard Frobenius map of $\mathrm{GL}_n(\mathbf{F})$, $(a_{ij}) \mapsto (a_{ij}^q)$.

Then $\mathrm{GL}_n(\mathbf{F})^F = \mathrm{GL}_n(q)$, $\mathrm{SL}_n(\mathbf{F})^F = \mathrm{SL}_n(q)$, $\mathrm{SO}_{2m+1}(\mathbf{F})^F = \mathrm{SO}_{2m+1}(q)$.

All groups of Lie type, except the Suzuki and Ree groups can be obtained in this way by a **standard** Frobenius map.

Sometimes it is easier to construct the groups by a non-standard Frobenius map.

EXAMPLE: THE UNITARY GROUPS

Let q be a power of p and let $\mathbf{G} := \mathrm{GL}_n(\mathbf{F})$. Let F denote the map

$$(a_{ij}) \mapsto \left(\left(a_{ij}^q\right)^{-1}\right)^{tr}.$$

Then F is a Frobenius map of $\mathbf{G}$, as $F^2 = F_{q^2}$.

In particular, $\mathbf{G}^F \leq \mathrm{GL}_n(\mathbb{F}_{q^2})$.

We have

$$F((a_{ij})) = (a_{ij}) \Leftrightarrow (a_{ij})^{tr}(a_{ij}^q) = I_n.$$

Thus, $\mathbf{G}^F$ is the unitary group of $\mathbb{F}_{q^2}^n$ with respect to the hermitian form $\langle (x_1, \ldots, x_n)^{tr}, (y_1, \ldots, y_n)^{tr} \rangle = \sum_{i=1}^n x_i y_i^q$.

In the following, $(\mathbf{G}, F)$ denotes a finite reductive group over $\mathbf{F}$.

THE LANG-STEINBERG THEOREM

THEOREM (LANG-STEINBERG, 1956/1968)

If $\mathbf{G}$ *is connected, the map* $\mathbf{G} \to \mathbf{G}$, $g \mapsto g^{-1}F(g)$ *is surjective.*

The assumption that $\mathbf{G}$ is connected is crucial here.

EXAMPLE

Let $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$, and $F : (a_{ij}) \mapsto (a_{ij}^q)$, q a power of p.

Then there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{G}$ such that

$$\begin{bmatrix} a^q & b^q \\ c^q & d^q \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

The Lang-Steinberg theorem is used to derive structural properties of $\mathbf{G}^F$.

MAXIMAL TORI AND THE WEYL GROUP

A torus of $\mathbf{G}$ is a closed subgroup isomorphic to $\mathbf{F}^* \times \cdots \times \mathbf{F}^*$.

A torus is maximal, if it is not contained in any larger torus of $\mathbf{G}$.

Crucial fact: Any two maximal tori of $\mathbf{G}$ are conjugate.

DEFINITION

The Weyl group W of $\mathbf{G}$ is defined by $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$.

EXAMPLE

Let $\mathbf{G} = \mathrm{GL}_n(\mathbf{F})$ and $\mathbf{T}$ the group of diagonal matrices. Then:

1. $\mathbf{T}$ is a maximal torus of $\mathbf{G}$,
2. $N_{\mathbf{G}}(\mathbf{T})$ is the group of monomial matrices,
3. $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ can be identified with the group of permutation matrices, i.e. $W \cong S_n$.

MAXIMAL TORI OF FINITE REDUCTIVE GROUPS

A maximal torus of $(\mathbf{G}, F)$ is a finite reductive group $(\mathbf{T}, F)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$.

A maximal torus of $G = \mathbf{G}^F$ is a subgroup T of the form $T = \mathbf{T}^F$ for some maximal torus $(\mathbf{T}, F)$ of $(\mathbf{G}, F)$.

EXAMPLE

A Singer cycle is an irreducible cyclic subgroup of $\mathrm{GL}_n(q)$ of order $q^n - 1$. This is a maximal torus of $\mathrm{GL}_n(q)$.

The maximal tori of $(\mathbf{G}, F)$ are classified (up to conjugation in G) by F-conjugacy classes of W.

These are the orbits under the action $v.w \mapsto vwF(v)^{-1}$, $v, w \in W$.

THE CLASSIFICATION OF MAXIMAL TORI

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$.

Let $w \in W$, and $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$ with $w = \dot{w}\mathbf{T}$.

By the Lang-Steinberg theorem, there is $g \in \mathbf{G}$ such that $\dot{w} = g^{-1}F(g)$.

One checks that ${}^{g}\mathbf{T}$ is F-stable, and so $({}^{g}\mathbf{T}, F)$ is a maximal torus of $(\mathbf{G}, F)$.

The map $w \mapsto ({}^{g}\mathbf{T}, F)$ induces a bijection between the set of F-conjugacy classes of W and the set of G-conjugacy classes of maximal tori of $(\mathbf{G}, F)$.

We say that ${}^{g}\mathbf{T}$ is obtained from $\mathbf{T}$ by twisting with w.

THE MAXIMAL TORI OF $\mathrm{GL}_n(q)$

Let $\mathbf{G} = \mathrm{GL}_n(\mathbf{F})$ and $F = F_q$ a standard Frobenius morphism.

Then F acts trivially on $W = S_n$, i.e. the maximal tori of $G = \mathrm{GL}_n(q)$ are parametrized by partitions of n.

If $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition of n, we write T_λ for the corresponding maximal torus.

We have

$$|T_\lambda| = (q^{\lambda_1} - 1)(q^{\lambda_2} - 1) \cdots (q^{\lambda_l} - 1).$$

Each factor $q^{\lambda_i} - 1$ of $|T_\lambda|$ corresponds to a cyclic direct factor of T_λ of this order.

A representative for T_λ can be obtained by taking a Singer cycle of $\mathrm{GL}_{\lambda_i}(q)$, $1 \leq i \leq l$, and embedding $\mathrm{GL}_{\lambda_1}(q) \times \ldots \times \mathrm{GL}_{\lambda_l}(q)$ diagonally into G.

THE STRUCTURE OF THE MAXIMAL TORI

Let $\mathbf{T}'$ be an F-stable maximal torus of $\mathbf{G}$, obtained by twisting the reference torus $\mathbf{T}$ with $w = \dot{w}\mathbf{T} \in W$.

I.e. there is $g \in \mathbf{G}$ with $g^{-1}F(g) = \dot{w}$ and $\mathbf{T}' = {}^g\mathbf{T}$.
Then

$$T' = (\mathbf{T}')^F \cong \mathbf{T}^{wF} := \{t \in \mathbf{T} \mid t = \dot{w}F(t)\dot{w}^{-1}\}.$$

Indeed, for $t \in \mathbf{T}$ we have $gtg^{-1} = F(gtg^{-1})$ [$= F(g)F(t)F(g)^{-1}$] if and only if $t \in \mathbf{T}^{wF}$.

EXAMPLE

Let $\mathbf{G} = \mathrm{GL}_n(\mathbf{F})$, and $\mathbf{T}$ the group of diagonal matrices.
Let $w = (1, 2, \ldots, n)$ be an n-cycle. Then

$$\mathbf{T}^{wF} = \{\mathrm{diag}[t, t^q, \ldots, t^{q^{n-1}}] \mid t \in \mathbf{F}, t^{q^n-1} = 1\},$$

and so $\mathbf{T}^{wF}$ is cyclic of order $q^n - 1$.

BN-PAIRS

This axiom system was introduced by Jaques Tits to allow a uniform treatment of groups of Lie type.

DEFINITION

The subgroups B and N of the group G form a BN-pair, if:

1. $G = \langle B, N\rangle$;
2. $T := B \cap N$ *is normal in* N;
3. $W := N/T$ *is generated by a set* S *of involutions;*
4. *If* $\dot{s} \in N$ *maps to* $s \in S$ *(under* $N \to W$*), then* $\dot{s}B\dot{s} \neq B$;
5. *For each* $n \in N$ *and* $\dot{s}$ *as above,* $(B\dot{s}B)(BnB) \subseteq B\dot{s}nB \cup BnB$.

W is called the Weyl group of the *BN*-pair G. It is a Coxeter group with Coxeter generators S.

COXETER GROUPS

Let $(m_{ij})_{1 \leq i,j \leq r}$ be a symmetric matrix with $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ satisfying $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$.

The group

$$W := W(m_{ij}) := \left\langle s_1, \ldots, s_r \mid (s_i s_j)^{m_{ij}} = 1 (i \neq j), s_i^2 = 1 \right\rangle_{\text{group}},$$

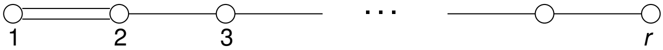
is called the Coxeter group of (m_{ij}), the elements $s_1, \ldots, s_r$ are the Coxeter generators of W.

The relations $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$ are called the braid relations.

In view of $s_i^2 = 1$, they can be written as $s_i s_j s_i \cdots = s_j s_i s_j \cdots$

The matrix (m_{ij}) is usually encoded in a Coxeter diagram, e.g.

B_r:



with number of edges between nodes $i \neq j$ equal to $m_{ij} - 2$.

THE *BN*-PAIR OF $\mathrm{GL}_n(k)$ AND OF $\mathrm{SO}_n(k)$

Let k be a field and $G = \mathrm{GL}_n(k)$. Then G has a BN-pair with:

- B: group of upper triangular matrices;
- N: group of monomial matrices;
- $T = B \cap N$: group of diagonal matrices;
- $W = N/T \cong S_n$: group of permutation matrices.

Let n be odd and let $\mathrm{SO}_n(k) = \{g \in \mathrm{SL}_n(k) \mid g^{tr} J g = J\}$ be the orthogonal group.

If B, N are as above, then

$$B \cap \mathrm{SO}_n(k), N \cap \mathrm{SO}_n(k)$$

is a BN-pair of $\mathrm{SO}_n(k)$.

SPLIT BN-PAIRS OF CHARACTERISTIC p

Let G be a group with a BN-pair (B, N).

This is said to be a split BN-pair of characteristic p, if the following additional hypotheses are satisfied:

6. $B = UT$ with $U = O_p(B)$, the largest normal p-subgroup of B, and T a complement of U.
7. $\bigcap_{n \in N} nBn^{-1} = T$. (Recall $T = B \cap N$.)

EXAMPLES

1. A semisimple algebraic group over $\mathbf{F}$ and a finite reductive group of characteristic p have split BN-pairs of characteristic p.

2. If $G = \mathrm{GL}_n(\mathbf{F})$ or $\mathrm{GL}_n(q)$, q a power of p, then U is the group of upper triangular unipotent matrices.

   In the latter case, U is a Sylow p-subgroup of G.

PARABOLIC SUBGROUPS AND LEVI SUBGROUPS

Let G be a group with a split BN-pair of characteristic p.

Any conjugate of B is called a Borel subgroup of G.

A parabolic subgroup of G is one containing a Borel subgroup.

Let $P \leq G$ be a parabolic subgroup. Then

$$P = UL$$

with

- $U = O_p(P)$ is the largest normal p-subgroup of P.
- L is a complement to U in P.

This is called a Levi decomposition of P, and L is a Levi subgroup of G.

A Levi subgroup is itself a group with a split BN-pair of characteristic p.

EXAMPLES FOR PARABOLIC SUBGROUPS, I

In classical groups, parabolic subgroups are the stabilisers of isotropic subspaces.

Let $G = \mathrm{GL}_n(q)$, and $(\lambda_1, \ldots, \lambda_l)$ a partition of n. Then

$$P = \left\{ \begin{bmatrix} \mathrm{GL}_{\lambda_1}(q) & \star & \star \\ & \ddots & \star \\ & & \mathrm{GL}_{\lambda_l}(q) \end{bmatrix} \right\}$$

is a typical parabolic subgroup of G. A corresponding Levi subgroup is

$$L = \left\{ \begin{bmatrix} \mathrm{GL}_{\lambda_1}(q) & & \\ & \ddots & \\ & & \mathrm{GL}_{\lambda_l}(q) \end{bmatrix} \right\} \cong GL_{\lambda_1}(q) \times \cdots \times \mathrm{GL}_{\lambda_l}(q).$$

$B = UT$ with T the diagonal matrices and U the upper triangular unipotent matrices is a Levi decomposition of B.

EXAMPLES FOR PARABOLIC SUBGROUPS, II

Let $G = \mathrm{SO}_{2m+1}(q)$. Every Levi subgroup of G is conjugate to one of the form

$$L = \left\{ \begin{bmatrix} A & & \\ & B & \\ & & A^* \end{bmatrix} \mid A \in M, B \in \mathrm{SO}_{2l+1}(q) \right\} \cong M \times \mathrm{SO}_{2l+1}(q),$$

where M is a Levi subgroup of $\mathrm{GL}_{m-l}(q)$, and $A^* = J\left(A^{-1}\right)^{tr} J$.

A parabolic subgroup P containing L is $P = UL$ with

$$U = \left\{ \begin{bmatrix} I_{m-l} & \star & \star \\ & I_{2l+1} & \star \\ & & I_{m-l} \end{bmatrix} \right\} \leq \mathrm{SO}_{2m+1}.$$

The structure of a Levi subgroup of G very much resembles the structure of G.

End of Lecture I.

Thank you for your attention!