REPRESENTATION THEORY FOR GROUPS OF LIE TYPE

LECTURE II: HARISH-CHANDRA PHILOSOPHY

Gerhard Hiss

Lehrstuhl D für Mathematik
RWTH Aachen University

HIM Summer School
Logic and Algorithms in Group Theory
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Contents

REPRESENTATIONS: DEFINITIONS

Let G be a group and k a field.

A k-representation of G is a homomorphism $X : G \to \mathrm{GL}(V)$, where V is a k-vector space. (X is also called a representation of G on V.)

If $d := \dim_k(V)$ is finite, d is called the degree of X.

X reducible, if there exists a G-invariant subspace $0 \neq W \neq V$ (i.e. $X(g)(w) \in W$ for all $w \in W$ and $g \in G$).

In this case we obtain a sub-representation of G on W and a quotient representation of G on V/W.

Otherwise, X is called irreducible.

There is a natural notion of equivalence of k-representations.

COMPOSITION SERIES

Let X be a k-representation of G on V with $\dim V < \infty$.

Consider a chain $\{0\} < V_1 < \cdots < V_l = V$ of G-invariant subspaces, such that the representation X_i of G on V_i/V_{i-1} is irreducible for all $1 \leq i \leq l$.

Choosing a basis of V through the V_i, we obtain a matrix representation $\tilde{X}$ of G, equivalent to X, s.t.:

$$\tilde{X}(g) = \begin{bmatrix} X_1(g) & \star & \cdots & \star \\ 0 & X_2(g) & \cdots & \star \\ 0 & 0 & \ddots & \star \\ 0 & 0 & \cdots & X_l(g) \end{bmatrix} \quad \text{for all } g \in G.$$

The X_i (or the V_i/V_{i-1}) are called the irreducible constituents (or composition factors) of X (or of V).

They are unique up to equivalence and ordering.

MODULES AND THE GROUP ALGEBRA

Let $X : G \to \mathrm{GL}(V)$ be a k-representation of G on V.

For $v \in V$ and $g \in G$, write $g.v := X(g)(v)$.
This makes V into a left kG-module.

Here, kG denotes the group algebra of G over k:

$$kG := \left\{ \sum_{g \in G} a_g g \mid a_g \in k, a_g = 0 \text{ for almost all } g \right\},$$

with multiplication inherited from G.

- X is irreducible if and only if V is a simple kG-module.
- X and $Y : G \to \mathrm{GL}(W)$ are equivalent, if and only if V and W are isomorphic as kG-modules.

Classification of representations

Fact

If G is finite, there are only finitely many irreducible k-representations of G up to equivalence.

In view of the classification of the finite simple groups, the following principal goal is very natural.

Objective

"Classify" all irreducible representations of all finite simple groups.

More specifically, find **labels** for their irreducible representations, find the **degrees** of these, etc.

"Most" finite simple groups are groups of Lie type. So let's concentrate on these.

Three cases

In the following, let $G = \mathbf{G}^F$ be a finite reductive group.

Recall that $\mathbf{G}$ is a connected reductive algebraic group over $\mathbf{F}$, $\mathrm{char}(\mathbf{F}) = p$, and that F is a Frobenius morphism of $\mathbf{G}$.

Let k be algebraically closed.

It is natural to distinguish three cases:

1. $\mathrm{char}(k) = p$ (usually $k = \mathbf{F}$); defining characteristic
2. $\mathrm{char}(k) = 0$; ordinary representations
3. $0 < \mathrm{char}(k) \neq p$; non-defining characteristic

In this series of lectures, I will mainly talk about Case 2.

From now on, assume that all kG-modules are finite-dimensional.

Harish-Chandra induction

View G as a finite group with a split BN-pair of characteristic p.

Let L be a Levi subgroup of G, and M a kL-module.

Let P be a parabolic subgroup with Levi complement L. Write $\widetilde{M}$ for the inflation of M to P.

Put

$$\begin{aligned} R^G_{L\subset P}(M) &:= \mathrm{Hom}_{kP}(kG, \widetilde{M}) \\ &= \{f : kG \to M \mid a.f(b) = f(ab) \quad \forall a \in P, b \in kG\}. \end{aligned}$$

$R^G_{L\subset P}(M)$ is a kG-module, a Harish-Chandra induced module.

Action of G: $[g.f](b) := f(bg)$, $g \in G$, $b \in kG$, $f \in R^G_{L\subset P}(M)$.

Remarks on Harish-Chandra induction

The module $R^G_{L\subset P}(M) = \mathrm{Hom}_{kP}(kG, \widetilde{M})$ is a coinduced module.

This definition is due to Harish-Chandra and is inspired by the notion of modular forms.

$R^G_{L\subset P}(M)$ is (naturally) isomorphic to the induced module $kG \otimes_{kP} \widetilde{M}$.

Theorem

If p is invertible in k, then $R^G_{L\subset P}(M)$ is independent of the choice of P with Levi complement L.

- Lusztig, 1970s: k a field of characteristic 0
- Dipper-Du, 1993: k a field of characteristic $\neq p$
- Howlett-Lehrer, 1994: k not necessarily a field.

An example: $\mathrm{GL}_3(q)$

Let $G = \mathrm{GL}_3(q)$, where q is a power of p,

$$L = \left\{ \left[\begin{array}{cc|c} \star & \star & 0 \\ \star & \star & 0 \\ \hline 0 & 0 & \star \end{array} \right] \right\},$$

$$P = \left\{ \left[\begin{array}{cc|c} \star & \star & \star \\ \star & \star & \star \\ \hline 0 & 0 & \star \end{array} \right] \right\}, \text{ and } P' = \left\{ \left[\begin{array}{cc|c} \star & \star & 0 \\ \star & \star & 0 \\ \hline \star & \star & \star \end{array} \right] \right\}.$$

Then $R^G_{L \subset P}(M) \cong R^G_{L \subset P'}(M)$ for all kL-modules M.

Notice that P and P' are **not** conjugate in G.

CENTRALISER ALGEBRAS

From now on we suppress the P from the notation for Harish-Chandra induction, i.e. we write R_L^G for $R_{L\subset P}^G$.

With L and M as before, we write

$$\mathcal{H}(L, M) := \mathrm{End}_{kG}(R_L^G(M)).$$

for the endomorphism ring of $R_L^G(M)$.

$\mathcal{H}(L, M)$ is also called the centraliser algebra or Hecke algebra of $R_L^G(M)$.

$\mathcal{H}(L, M)$ is used to analyse the submodules and quotients of $R_L^G(M)$ via Fitting correspondence.

THE FITTING CORRESPONDENCE

Let A be a ring, X an A-module and $E := \mathrm{End}_A(X)$.

PROPOSITION (FITTING CORRESPONDENCE)

Suppose that $X = X_1 \oplus \cdots \oplus X_n$ is a direct decomposition of X into A-submodules X_i.
Put $E_i := \mathrm{Hom}_A(X, X_i)$, $1 \leq i \leq n$, viewed as a subset of E.
Then the following hold:

1. *The E_i are (right) ideals of E and $E = E_1 \oplus \cdots \oplus E_n$.*
2. *$E_i \cong E_j$ as E-modules if and only if $X_i \cong X_j$ as A-modules.*
3. *E_i is indecomposable as an E-module if and only if X_i is indecomposable as an A-module.*

This is an important link between the structure of X and that of E.

COXETER GROUPS: RECOLLECTION

Recall that the Weyl group of G (as group with BN-pair) is a Coxeter group.

Let $(m_{ij})_{1\leq i,j\leq r}$ be a symmetric matrix with $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ satisfying $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$.

The group

$$W := W(m_{ij}) := \left\langle s_1, \ldots, s_r \mid (s_i s_j)^{m_{ij}} = 1 (i \neq j), s_i^2 = 1 \right\rangle_{\text{group}},$$

is called the Coxeter group of M, the elements $s_1, \ldots, s_r$ are the Coxeter generators of W.

The relations $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$ are called the braid relations.

In view of $s_i^2 = 1$, they can be written as $s_i s_j s_i \cdots = s_j s_i s_j \cdots$

THE IWAHORI-HECKE ALGEBRA

Let W be a Coxeter group with Coxeter matrix (m_{ij}).

Let $\mathbf{v} = (v_1, \ldots, v_r) \in k^r$ with $v_i = v_j$, whenever s_i and s_j are conjugate in W.

The algebra

$$\mathcal{H}_{k,\mathbf{v}}(W) := \left\langle T_{s_1}, \ldots, T_{s_r} \mid T_{s_i}^2 = v_i 1 + (v_i - 1)T_{s_i}, \text{ braid rel's } \right\rangle_{k\text{-alg.}}$$

is the Iwahori-Hecke algebra of W over k with parameter $\mathbf{v}$.

Braid rel's: $T_{s_i}T_{s_j}T_{s_i}\cdots = T_{s_j}T_{s_i}T_{s_j}\cdots$ (m_{ij} factors on each side)

FACT

$\mathcal{H}_{k,\mathbf{v}}(W)$ is a free k-algebra with k-basis T_w, $w \in W$.

Note that $\mathcal{H}_{k,(1,\ldots,1)}(W) \cong kW$, so that $\mathcal{H}_{k,\mathbf{v}}(W)$ is a deformation of the group algebra kW.

THE THEOREM OF IWAHORI AND MATSUMOTO

Let $k[G/B]$ denote the permutation module on G/B.

This is a special case of a Harish-Chandra induced module, i.e. $k[G/B] = R_T^G(\mathbf{1})$, where $\mathbf{1}$ denotes the trivial kT-module.

Put $E := \mathrm{End}_{kG}(k[G/B]) = \mathcal{H}(T, \mathbf{1})$.

THEOREM (IWAHORI/MATSUMOTO)

E is the Iwahori-Hecke algebra of W over k with parameter $(q_i = [B \colon {}^{s_i}B \cap B])_{1 \leq i \leq r}$.

HARISH-CHANDRA CLASSIFICATION

From now on let k be an algebraically closed field with $\mathrm{char}(k) \neq p$.

A simple kG-module V is called cuspidal, if V is **not** a **submodule** of $R_L^G(M)$ for some **proper** Levi subgroup L of G.

Harish-Chandra philosophy (HC-induction, cuspidality) yields the following classification.

THEOREM (HARISH-CHANDRA (1968), LUSZTIG ('70S) (CHAR$(k) = 0$), GECK-H.-MALLE (1996) (CHAR$(k) > 0$))

$$\{V \mid V \text{ simple } kG\text{-module}\} / \text{isomorphism}$$
$$\updownarrow$$
$$\left\{ (L, M, \theta) \;\middle|\; \begin{array}{c} L \text{ Levi subgroup of } G \\ M \text{ simple, cuspidal } kL\text{-module} \\ \theta \text{ simple } \mathcal{H}(L, M)\text{-module} \end{array} \right\} / \text{conjugacy}$$

Main steps in Harish-Chandra classification, I

Let V be a simple kG-module.

Let L be a Levi subgroup of minimal order such that $V \leq R_L^G(M)$ for some kL-module M of minimal dimension.

Then M is simple since R_L^G is exact.

Moreover, M is cuspidal since Harish-Chandra induction is transitive and exact.

The pair (L, M) is uniquely determined from V up to conjugation in G.

MAIN STEPS IN HARISH-CHANDRA CLASSIFICATION, II

$R_L^G(M)$ is a direct sum of indecomposable kG-modules (components), each having a unique simple submodule.

These components are determined by their simple submodules up to isomorphism.

Thus $V \leq R_L^G(M)$ determines an isomorphism type of components of $R_L^G(M)$.

By Fitting correspondence, the simple modules of $\mathcal{H}(L, M)$ are in bijection to the isomorphism types of components of $R_L^G(M)$.

HARISH-CHANDRA SERIES

DEFINITION

Two simple kG-modules V and V' are said to lie in the same Harish-Chandra series, if V and V' determine the same cuspidal pair (L, M).

*In other words, if V **and** V' are submodules of $R_L^G(M)$ for some cuspidal kL-module M of some Levi subgroup L.*

Let $\mathcal{E}(L, M)$ denote the Harish-Chandra series determined by the cuspidal pair (L, M).

Remarks: The set of simple kG-modules (up to isomorphism) is partitioned into Harish-Chandra series.

The elements of $\mathcal{E}(L, M)$ are in bijection with the simple modules of $\mathcal{H}(L, M)$.

PROBLEMS IN HARISH-CHANDRA PHILOSOPHY

The above classification theorem leads to the three tasks:

1. Determine the cuspidal pairs (L, M).
2. For each of these, "compute" $\mathcal{H}(L, M)$.
3. Classify the simple $\mathcal{H}(L, M)$-modules.

State of the art in case $\text{char}(k) = 0$ (Lusztig):

- Cuspidal simple kG-modules arise from étale cohomology groups of Deligne-Lusztig varieties.
- $\mathcal{H}(L, M)$ is an Iwahori-Hecke algebra (see above), corresponding to the Coxeter group $W_G(L, M)$ (see below).
- $\mathcal{H}(L, M) \cong kW_G(L, M)$ (Tits deformation theorem).

THE RELATIVE WEYL GROUP

Let L be a Levi subgroup of G. The group $W_G(L) := (N_G(L) \cap N)L/L$ is the relative Weyl group of L.

Here, N is the N from the BN-pair of G.

It is introduced to avoid trivialities: If $G = \mathrm{GL}_n(2)$, and $L = T$ is the torus of diagonal matrices, then $L = \{1\}$ and $N_G(L) = G$.

Alternative definition: $W_G(L) = N_G(\mathbf{L})/L$.

$W_G(L)$ is naturally isomorphic to a subgroup of W.

If M is a kL-module, $W_G(L, M) := \{w \in W_G(L) \mid {}^wM \cong M\}$.

Example: $\mathrm{SL}_2(q)$

Let $G = \mathrm{SL}_2(q)$ and $\mathrm{char}(k) = 0$.

The group T of diagonal matrices is the only proper Levi subgroup; it is a cyclic group of order $q - 1$.

We have $W = W_G(T) = \langle T, s \rangle / T$ with $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Let M be a simple kT-module. Then $\dim M = 1$ and M is cuspidal, and $\dim R_T^G(M) = q + 1$ (since $[G : B] = q + 1$).

Case 1: $W_G(T, M) = \{1\}$. Then $\mathcal{H}(T, M) \cong k$ and $R_T^G(M)$ is simple of dimension $q + 1$.

Case 2: $W_G(T, M) = W_G(T)$. Then $\mathcal{H}(T, M) \cong kW_G(T)$, and $R_T^G(M)$ is the sum of two non-isomorphic simple kG-modules.

STATE OF THE ART IN CASE CHAR$(k) \neq 0$

Suppose that $\mathrm{char}(k) > 0$ (and $\neq p$).

- $\mathcal{H}(L, M)$ is a "twisted" "Iwahori-Hecke algebra" corresponding to an "extended" Coxeter group (Howlett-Lehrer (1980), Geck-H.-Malle (1996)), namely $W_G(L, M)$;
  parameters of $\mathcal{H}(L, M)$ not known in general.
- $G = \mathrm{GL}_n(q)$; everything known (Dipper-James, 1980s)
- G classical group, $\mathrm{char}(k)$ "linear"; everything known (Gruber-H., 1997).
- In general, classification of cuspidal pairs open.

End of Lecture II.

Thank you for your listening!