Computational Representation
Theory of Finite Groups

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Throughout my lecture, $G$ denotes a finite group and $K$ a field.
A $K$-representation of $G$ of degree $d$ is a homomorphism

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$$\chi : G \to \text{GL}(V),$$

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Choosing a basis of $V$, we obtain a matrix representation $G \to \text{GL}_d(K)$ to compute with.
Representations: Classification

- There are only finitely many irreducible \( K \)-representations of \( G \) up to equivalence.
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- There are only finitely many irreducible $K$-representations of $G$ up to equivalence.
- Classify all irreducible representations of $G$.
- Describe all irreducible representations of all finite simple groups.
- Use a computer for sporadic simple groups.
Representations: Constructions

Representations can be constructed
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– from permutation representations,
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– from two representations through their Kronecker product,

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– in various other ways.
A permutation representation of $G$ on the finite set $\Omega = \{\omega_1, \ldots, \omega_n\}$ is a homomorphism

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Replacing each $\kappa(g) \in S_\Omega$ by the corr. linear map $\chi(g)$ of $K\Omega$ (permuting its basis as $\kappa(g)$), we obtain a $K$-representation of $G$. 
Invariant Subspaces

Let $\mathcal{X} : G \rightarrow \text{GL}(V)$ be a $K$-representation of $G$. 

Let $W$ be a $G$-invariant subspace of $V$, i.e.:

$w : g \in W$ for all $w \in W, g \in G$.

We obtain $K$-representations $\mathcal{X}_W : G \rightarrow \text{GL}(W)$ and $\mathcal{X}_{V=W} : G \rightarrow \text{GL}(V=W)$ in the natural way.
Invariant Subspaces

Let $\mathcal{X} : G \rightarrow \text{GL}(V)$ be a $K$-representation of $G$. For $v \in V$ and $g \in G$, write $v.g := v \cdot \mathcal{X}(g)$. 
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one obtains all irreducible representations of $G$. 
The Meat-Axe

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Since then it has been improved and enhanced by many people, including Derek Holt, Gábor Ivanyos, Klaus Lux, Jürgen Müller, Sarah Rees, and Michael Ringe.
The Meat-Axe: Basic Problems

How does one find a non-trivial proper $G$-invariant subspace of $V$?
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- Indeed, given $0 \neq w \in W$, the orbit \( \{w.g \mid g \in G\} \) spans a $G$-invariant subspace contained in $W$. 
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How does one prove that $\mathcal{X}$ is irreducible?
Let $A_1, \ldots, A_l$, be $(d \times d)$-matrices over $K$. 
Norton’s Irreducibility Criterion

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Put $\mathfrak{A} := K[A_1, \ldots, A_l]$ (algebra span).
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Write $A^t$ for the transpose of $A$, and $\mathfrak{A}^t := K[A_1^t, \ldots, A_l^t]$. 
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Let $B \in \mathcal{A}$.

Then one of the following occurs:
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3. Every non-trivial vector in the (left) nullspace of $B^t$ lies in a proper $A^t$-invariant subspace.

4. $A$ acts irreducibly on $K^{1 \times d}$. 
If $G = \langle g_1, \ldots, g_l \rangle$, put $A_i := \mathcal{X}(g_i)$, $1 \leq i \leq l$. 

Find singular $B$ with nullspace $N$ of small dimension (preferably 1).

For all $0 \neq w \in N$ test if $w : A = 0$. (Note that $w : A$ is $G$-invariant.) If YES for one $0 \neq w$ in the nullspace of $B$ test if $w : A^t = 0$. If YES, $X$ is irreducible.
The Meat-Axe: Basic Strategy

If \( G = \langle g_1, \ldots, g_l \rangle \), put \( A_i := \mathcal{X}(g_i), 1 \leq i \leq l \).

Find singular \( B \in \mathcal{A} \) (by a random search) with nullspace \( N \) of small dimension (preferably 1).
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For all $0 \neq w \in N$ test if $w \mathcal{A} = K^{1 \times d}$. (Note that $w \mathcal{A}$ is $G$-invariant.)
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For one $0 \neq w$ in the nullspace of $B^t$ test if $w.A^t = K^{1 \times d}$. If YES, $\mathcal{X}$ is irreducible.
The Meat-Axe: Remarks

The above strategy works very well if $K$ is small.
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As $K$ gets larger, it gets harder to find a suitable $B$ by a random search.

Holt and Rees use characteristic polynomials of elements of $\mathcal{A}$ to find suitable $B$s and also to reduce the number of tests considerably.
Rob Wilson’s Atlas

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the representation of $M$ of degree 196,882 over $\mathbb{F}_2$ by Linton, Parker, Walsh, and Wilson.
Computations in the Monster

A matrix of $M \leq \text{GL}(196882, 2)$: 5GB memory
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Holmes and Wilson:

- maximal subgroups of $M$,
  e.g., $\text{PGL}(2, 29)$ (2002), $\text{PSL}(2, 59)$ (2004)

- $\text{PSL}(2, 23)$, is **not** maximal (though in $M$)
Condensation

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To overcome this problem, Condensation is used (Thackray, Parker, ca. 1980).
Let $A$ be a $K$-algebra and $e \in A$ an idempotent,
Condensation: Theory

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Get a functor: $\text{mod-}A \to \text{mod-}eAe$, $M \mapsto Me$. 
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If $S \in \text{mod-} A$ is simple, then $Se = 0$ or simple.
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($A$ and $eAe$ have the same representations.)
Let $H \leq G$ with $\text{char}(K) \nmid |H|$. 
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\[
e := \frac{1}{|H|} \sum_{h \in H} h \in KG.
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Let $M := K\Omega$ be the permutation module w.r.t. an action of $G$ on the finite set $\Omega$. 

Then $Me$ is the set of $H$-fixed points in $M$. 
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For $g \in G$, need to describe action of $ege$ on $Me$. 
Let $\Omega_1, \ldots, \Omega_m$ be the $H$-orbits on $\Omega$. 
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The orbits sums $\widehat{\Omega}_j \in K\Omega$ form a basis of $Me$. 
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$$a_{ij} = \frac{1}{|\Omega_j|} |\Omega_i g \cap \Omega_j|.$$
Condensation: History

\[ H \subseteq G \]

\[ \text{Hake } H \times H \text{ in } F_{-} \text{ mult. as in } F_{-} \]

\[ \text{Pres double cases } H \times H \]

\[ \text{New multiplication } \]

\[ H \times H, H \times H = H \times (H \times H) \]

\[ \sigma_H = \max \sigma \left( \frac{2}{H \times H} \right) \]

\[ \sigma_H (x \times y) = \sigma (H \times (H \times H)) \]

\[ \text{Use this to define } x. \]
Condensation: History

$H \leq G$

H x H in $F_{\text{mul}}$ as in $F_{\text{mod}}$

Porder double cosets $H x H$

New multiplicity

$H x H \cdot H y H = H x (H y H)$

$\sigma_H = \text{max} \{ \sigma_H(x H y H) \}$

$\sigma_H(x y) = \sigma(H x H y H)$

Use H1 to define x.

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Condensation: Applications


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\[ \dim(M) = 976,841,775, \quad \dim(Me) = 1,403. \]

\[ \dim(M) = 1,113,229,656. \]
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$|\Omega_i g \cap \Omega_j|$ structure constants of $\mathcal{B}$, the intersection numbers of $\mathcal{G}$

($\mathcal{O}_j$ orbits of $H := \text{Stab}(\omega_1)$ on $\Omega$)
Foulkes’ Conjecture

Let $m \geq n > 0$ be integers.
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Let \( m \geq n > 0 \) be integers.

\[ S_m \lhd S_n \leq S_{mn} \quad \text{and} \quad S_n \lhd S_m \leq S_{mn}. \]
Foulkes’ Conjecture

Let $m \geq n > 0$ be integers.

$S_m \mathfrak{l} S_n \leq S_{mn}$ and $S_n \mathfrak{l} S_m \leq S_{mn}$.

$\Omega_{m,n}$: set of cosets of $S_m \mathfrak{l} S_n$ in $S_{mn}$. 
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$\Omega_{m,n}$: set of cosets of $S_m \triangleleft S_n$ in $S_{mn}$.

Conjecture (Foulkes, 1950):

$\mathbb{Q}\Omega_{m,n} \leq \mathbb{Q}\Omega_{n,m}$, as $\mathbb{Q}S_{mn}$-modules.
Foulkes’ Conjecture: Black, List

Black, List, 1989:

- Define \( M_{m;n} \) a matrix of size \( j_n; m_j \).
- If \( M_{m;n} \) has maximal rank, then Foulkes’ conjecture holds.
- If \( M_{m;m} \) is invertible, then \( M_{m;n} \) has maximal rank for all \( n \).
- \( M_{2;2} \) and \( M_{3;3} \) are invertible.
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– define (0, 1)-matrix $M^{m,n}$ of size $|\Omega_{n,m}| \times |\Omega_{m,n}|$
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$M^{m,m}_m$ is an adjacency matrix of the action of $S_m^2$ on the cosets of $S_m \wr S_m$. 

Jacob, 2004: $M^{4,4}_4$ is invertible.

Müller, Neunhöffer, 2004: $M^{5,5}_5$ is singular.
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Use Condensation to compute intersection numbers.
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Ramanujan Graphs

A $k$-regular undirected graph $\Gamma$ with

$$\lambda(\Gamma) \leq 2\sqrt{k - 1},$$

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If the Bose-Mesner algebra is commutative, these eigenvalues are entries of its character table.
Example: $G = J_2$

$\Omega = G/H \text{ with } H = 2^{2+4} \cdot (3 \times S_3)$
**Example:** \( G = J_2 \)

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Character table of Bose-Mesner algebra:

<table>
<thead>
<tr>
<th>( J_2 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( A_4 )</th>
<th>( A_5 )</th>
<th>( A_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>192</td>
<td>96</td>
<td>192</td>
<td>12</td>
<td>32</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>1</td>
<td>-18</td>
<td>6</td>
<td>2</td>
<td>-3</td>
<td>12</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>1</td>
<td>-28</td>
<td>16</td>
<td>12</td>
<td>7</td>
<td>-8</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>1</td>
<td>0</td>
<td>-12</td>
<td>12</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>1</td>
<td>10</td>
<td>-2</td>
<td>-18</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>( \chi_6 )</td>
<td>1</td>
<td>6</td>
<td>6</td>
<td>-6</td>
<td>-3</td>
<td>-4</td>
</tr>
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Sporadic Ramanujan Graphs

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She found 358 Ramanujan graphs.
Thank you for your attention!