IMPRIMITIVE IRREDUCIBLE REPRESENTATIONS OF FINITE QUASISIMPLE GROUPS

Gerhard Hiss

Lehrstuhl D für Mathematik
RWTH Aachen University

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THE PROJECT

This is a joint project with William Husen and Kay Magaard.

PROJECT

Classify the pairs $(G, G \hookrightarrow \mathrm{SL}(V))$ such that

1. *G is a finite quasisimple group,*
2. *V a finite dimensional vector space over some field K,*
3. *$G \hookrightarrow \mathrm{SL}(V)$ is absolutely irreducible and imprimitive.*

DEFINITION

Let G be a finite group, K a field and V a KG-module.

V is imprimitive, if there is a decomposition $V = V_1 \oplus \cdots \oplus V_m$, $m > 1$, such that the V_i are permuted by the action of G.
We call $H := N_G(V_1)$ a block stabiliser.

(OUTER) TENSOR PRODUCTS

THEOREM (ASCHBACHER, 2000)

Let K be an algebraically closed field, let G_i be finite groups, and let V_i be finite-dimensional KG_i-modules for $i = 1, 2$.

Then the $K[G_1 \times G_2]$-module $V_1 \boxtimes_K V_2$ is primitive, if and only if V_i is a primitive KG_i-module for $i = 1, 2$.

The proof is trickier than one would expect.

EXAMPLE (I FORGOT, WHO TOLD ME THIS)

Let $G = J_2$ and $K = \mathbb{C}$ (and we replace modules by characters).

$\chi := \chi_2 = 14$ and $\psi := \chi_{18} = 225$ are primitive, but

$$\chi \cdot \psi = \mathrm{Ind}_H^G(6)$$

is imprimitive, where $H = 2^{2+4} \colon (3 \times S_3)$.

MOTIVATION I: MAXIMAL SUBGROUPS

Let $\mathrm{Cl}(V)$ be a finite classical group on the K-vector space V.

Let G be a finite quasisimple subgroup of $\mathrm{Cl}(V)$, such that $G \hookrightarrow \mathrm{Cl}(V)$ is absolutely irreducible.

When is $N_{\mathrm{Cl}(V)}(G)$ a maximal subgroup of $\mathrm{Cl}(V)$?

NO in general, if V is imprimitive, i.e. in Aschbacher class $\mathcal{C}_2$:

- P. KLEIDMAN AND M. LIEBECK, The subgroup structure of the finite classical groups, CUP, 1990.

Similar classification project of quasisimple groups in Aschbacher class $\mathcal{C}_4$ (tensor decomposable representations) is contained in:

- K. MAGAARD AND P. H. TIEP, Irreducible tensor products of representations of finite quasi-simple groups of Lie type, 1998.

MOTIVATION II: MATRIX GROUPS COMPUTATION

- Given an absolutely irreducible matrix group $G \leq \mathrm{GL}_n(K)$ for some finite field K, and some quasisimple (or nearly simple) group G, decide if G lies in Aschbacher class $\mathcal{C}_2$.
- If the isomorphism type of G is known, a table look-up in our lists might help to answer this question.
- To cover nearly simple groups, we would also have to give information about extensions of imprimitive modules to automorphism groups.

SPORADIC SIMPLE GROUPS

Complete list of examples for sporadic simple groups:

G	$\dim(V)$	$N_G(V_1)$	V_1	$\mathrm{char}(K)$
M_{11}	11 55	$A_6.2_3$ $3^2 : Q_8.2$	1_2 1_3	0, 5, 11
M_{12}	66 120	$A_6.2^2$ M_{11}	1_3 $10_2, 10_3$	0, 5, 11 0, 5
M_{22}	231	$2^4 : A_6$	$3_1, 3_2$	3
M_{24}	1 771	$2^6 : 3.S_6$	1_2	0, 5, 7, 11, 23
McL	9 625	$U_4(3)$	$35_1, 35_2$	0, 5, 7, 11
Co_2	1 288 000 2 095 875	$U_6(2) : 2$ $2^{10} : M_{22} : 2$	$560_1, 560_2$ $45_2, 45_4$	0, 5, 7, 23 0, 3, 5, 23

There are a few more examples for covering groups of these.

THE ALTERNATING GROUPS, $K = \mathbb{C}$

Again we replace modules by characters.

THEOREM (DRAGOMIR DJOKOVIĆ, JERRY MALZAN, 1976)

Suppose that $G = A_n$, $n \geq 10$, and let $\chi \in \mathrm{Irr}(G)$ be imprimitive. Then one of the following holds.

1. $n = m^2 + 1$ *and* $\chi = \mathrm{Res}_{A_n}^{S_n}(\zeta^\lambda)$ *with* $\lambda = (m+1, m^{m-1})$.
   Also, $\chi = \mathrm{Ind}_{A_{n-1}}^{G}(\chi_1)$ *with* χ_1 *a constituent of* $\mathrm{Res}_{A_{n-1}}^{S_{n-1}}(\zeta^\mu)$ *with* $\mu = (m^m)$.
2. $n = 2m$ *and* $\chi = \mathrm{Res}_{A_n}^{S_n}(\zeta^\lambda)$ *with* $\lambda = (m+1, 1^{m-1})$.
   Also, $\chi = \mathrm{Ind}_{N_{A_n}(S_m \times S_m)}^{G}(\chi_1)$ *with* $\chi_1(1) = 1$.

The classification for A_n is complete in all characteristics.

THE COVERING GROUPS OF THE ALTERNATING GROUPS, $K = \mathbb{C}$

THEOREM (DANIEL NETT, FELIX NOESKE, 2009)

Suppose that $G = 2.A_n$, $n \geq 10$, is the covering group of A_n, and let $\psi \in \mathrm{Irr}(G)$ be imprimitive.

Then $n = 1 + m(m+1)/2$, and $\psi = \mathrm{Res}^{2.S_n}_{2.A_n}(\sigma^\lambda)$ with $\lambda = (m+1, m-1, m-2, \ldots, 1)$.

Also, $\psi = \mathrm{Ind}^{2.A_n}_{2.A_{n-1}}(\psi_1)$ with ψ_1 a constituent of $\mathrm{Res}^{2.S_{n-1}}_{2.A_{n-1}}(\sigma^\mu)$ with $\mu = (m, m-1, \ldots, 1)$.

The classification for $2.A_n$ in positive characteristics is still open.

GROUPS OF LIE TYPE IN DEFINING CHARACTERISTICS

THEOREM (GARY SEITZ, 1988)

Let G be a finite quasisimple group of Lie type of characteristic p, and let K be an algebraically closed field with $\mathrm{char}(K) = p$.

Suppose that V is an irreducible, imprimitive KG-module.

Then G is one of

$$\mathrm{PSL}_2(5), \mathrm{PSL}_2(7), \mathrm{SL}_3(2), \mathrm{PSp}_4(3),$$

and V is the Steinberg module.

SOME EASY CHARACTERISTIC-FREE CRITERIA

Let G be a finite group, $H \leq G$, and K a field.
Suppose that H is the block stabiliser of an absolutely irreducible, imprimitive KG-module V. Then

1. $[G : H]$ divides $\dim_K(V)$.
2. $|H|^2 \geq |G|$.
3. For all $t \in G \setminus H$, the group ${}^tH \cap H$ is **not** centralised by t. In particular ${}^tH \cap H \neq \{1\}$ for all $t \in G$.
4. Suppose that $H = C_G(a)$ for some $a \in G$. Then $t \notin \langle {}^ta, a \rangle$ for all $t \in G \setminus H$.

Proof of 1: Clear, since $V = \mathrm{Ind}_H^G(V_1) = KG \otimes_{KH} V_1$.
Proof of 2: $[G : H]^2 \leq \dim_K(V)^2 \leq |G|$.
Proof of 3: This is a consequence of Mackey's theorem.
Proof of 4: For $t \in G$, ${}^tH \cap H = C_G({}^ta, a)$. Hence $t \notin \langle {}^ta, a \rangle$ for $t \in G \setminus H$, since such a t does not centralise ${}^tH \cap H$ by 3.

NON-PARABOLIC BLOCK STABILISERS

Large subgroups of groups of Lie type are **in general** parabolic subgroups.
There are, however, many exceptions, causing a lot of trouble.

EXAMPLE

Let $G = \mathrm{Sp}_{2m}(q)$ *with* m *even and* $q > 3$ *odd, and let*
$H = \langle H_0, s \rangle$ *with* $H_0 = \mathrm{Sp}_m(q) \times \mathrm{Sp}_m(q)$ *and* $s = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$.

Put $a = \begin{bmatrix} \alpha I_m & 0 \\ 0 & \alpha^{-1} I_m \end{bmatrix}$, *where* α *is a generator of* $\mathbb{F}_q^*$.
Then $H_0 = C_G(a)$.
Put $t := \begin{bmatrix} I_m & N \\ N & I_m \end{bmatrix}$ *with* $N := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
Then $t \in C_G(s) \setminus H$ *and* $t \in \langle {}^t a, a \rangle$, *hence* t *centralises* ${}^t H \cap H$.
In particular, H *is not the block stabiliser of an imprimitive irreducible* KG*-module.*

PARABOLIC BLOCK STABILISERS

Let G be a finite quasisimple group of Lie type of characteristic p, and let K be a field of characteristic $\neq p$.

THEOREM-CONJECTURE

Let $H \leq G$ be the block stabiliser of an absolutely irreducible, imprimitive KG-module. Then H is a parabolic subgroup of G.

PROPOSITION

Let P be a parabolic subgroup of G with unipotent radical U $(= O_p(P))$.

Let V_1 be a KP-module such that $\mathrm{Ind}_P^G(V_1)$ is irreducible. Then U is in the kernel of V_1.

In other words, $\mathrm{Ind}_P^G(V_1)$ is Harish-Chandra induced.

This allows to apply Harish-Chandra theory to our classification problem, reducing certain aspects to Weyl groups.

Asymptotics

By Harish-Chandra theory, a large proportion of absolutely irreducible modules of a group of Lie type are imprimitive.

Remark

Let L be a Levi complement of the parabolic subgroup P of G, and let V_1 be an irreducible KL-module which is rigid. This means, roughly, that the stabiliser of V_1 in $N_G(L)$ equals L.

Then $\mathrm{Ind}_P^G(\mathrm{Infl}_L^P(V_1))$ is irreducible.

Examples

(1) $G = \mathrm{GL}_n(q)$, $L = \mathrm{GL}_m(q) \times \mathrm{GL}_{n-m}(q)$ with $m \neq n - m$. Then every absolutely irreducible KL-module is rigid.

(2) Let $G = \mathrm{SL}_n(q)$, $K = \mathbb{C}$. Then $|Irr(G)| = q^{n-1} + O(q^{n-2})$. The number of imprimitive elements in $\mathrm{Irr}(G)$ equals $(1 - 1/n)q^{n-1} + O(q^{n-2})$.

EXAMPLE: $SL_2(q)$, q EVEN

	C_1	C_2	$C_3(a)$	$C_4(b)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(m)$	$q+1$	1	$\zeta^{am}+\zeta^{-am}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi^{bn}-\xi^{-bn}$

$a, m = 1, \ldots, (q-2)/2, \qquad b, n = 1, \ldots, q/2,$

The characters $\chi_3(m)$ are imprimitive, the others are primitive.

Number of irreducible characters: $q+1$.

Number of imprimitive irreducible characters: $q/2-1$.

THE CLASSIFICATION FOR $\mathrm{GL}_n(q)$

Let $G = \mathrm{GL}_n(q)$, $K = \mathbb{C}$. A unipotent character of G is an irreducible constituent of the permutation character on the cosets of a Borel subgroup of G (the group of upper triangular matries).
By Lusztig-theory, we have

$$\mathrm{Irr}(G) = \{\chi_{s,\lambda} \mid s \in G \text{ semisimple}, \lambda \in \mathrm{Irr}(C_G(s)) \text{ unipotent}\}.$$

Here, s has to be taken modulo conjugation in G.
Notice that

$$C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_k}(q^{d_k}).$$

THEOREM

$\chi_{s,\lambda} \in \mathrm{Irr}(G)$ is Harish-Chandra primitive if and only if the minimal polynomial of s is irreducible. In particular, every unipotent character is Harish-Chandra primitive.

DESCENT FROM $\mathrm{GL}_n(q)$ TO $\mathrm{SL}_n(q)$

The descent from $\mathrm{GL}_n(q)$ to $\mathrm{SL}_n(q)$ is not so easy to describe. Suppose that $K = \mathbb{C}$.

EXAMPLE (CÉDRIC BONNAFÉ)

Suppose that q is odd, let $G = \mathrm{GL}_4(q)$ and P a parabolic subgroup with Levi complement $L = \mathrm{GL}_2(q) \times \mathrm{GL}_2(q)$.

Let $\mathbf{1}$ denote the trivial character and $\mathbf{1}^-$ the unique linear character of $\mathrm{GL}_2(q)$ of order 2.

Then $\chi := \mathrm{Ind}_P^G(\mathrm{Infl}_L^P(\mathbf{1} \boxtimes \mathbf{1}^-))$ is irreducible, hence imprimitive.

However, $\mathrm{Res}^G_{\mathrm{SL}_4(q)}(\chi) = \psi_1 + \psi_2$, with irreducible, ***primitive*** *characters ψ_1, ψ_2.*

THEOREM

Let $\chi \in \mathrm{Irr}(\mathrm{GL}_n(q))$ be Harish-Chandra primitive.

Then $\mathrm{Res}^{\mathrm{GL}_n(q)}_{\mathrm{SL}_n(q)}(\chi)$ is irreducible and primitive.

Thank you for listening!