CONDENSATION
AN EXAMPLE: THE FISCHER GROUP $Fi_{23}$ MODULO 2

Computational Representation Theory – Lecture IV

Gerhard Hiss

Lehrstuhl D für Mathematik
RWTH Aachen University

First De Brún Workshop on Computational Algebra
Galway, 21 July – 1 August 2008
CONTENTS

1. Condensation
2. An Example: The Fischer Group $Fi_{23}$ Modulo 2
Throughout this lecture, $G$ denotes a finite group and $F$ a field.

Also, $\mathcal{A}$ denotes a finite-dimensional $F$-algebra, $J(\mathcal{A})$ the Jacobson radical of $\mathcal{A}$ (i.e., the intersection of the maximal right ideals of $\mathcal{A}$).

$\text{mod-}\mathcal{A}$: category of finite-dimensional right $\mathcal{A}$-modules
The MeatAxe can reduce representations of degree up to 50,000 over $\mathbb{F}_2$.

Over larger fields, only smaller degrees are feasible.

To overcome this problem, Condensation is used (Thackray, Parker, ca. 1980).
Condensation: Theory [J. A. Green 1980]

Let \( e \in \mathcal{A} \) an idempotent, i.e., \( 0 \neq e = e^2 \) (a projection).

Get exact functor: \( \text{mod-} \mathcal{A} \rightarrow \text{mod-} e \mathcal{A} e, \ V \mapsto Ve \).

If \( S \in \text{mod-} \mathcal{A} \) is simple, then \( Se = 0 \) or simple.

Let \( S_1, \ldots, S_n \) be the simple \( \mathcal{A} \)-modules (up to isomorphism).

Suppose that \( S_1 e \neq 0, \ldots, S_m e \neq 0, S_{m+1} e = \cdots = S_n e = 0 \).

Then \( S_1 e, \ldots, S_m e \) are exactly the simple \( e \mathcal{A} e \)-modules (up to isomorphism).

As the Condensation functor is exact, it sends a composition series of \( V \in \text{mod-} \mathcal{A} \) to a composition series of \( Ve \in \text{mod-} e \mathcal{A} e \).
If $Se \neq 0$ for all simple $S \in \text{mod-}\mathcal{A}$, then Condensation is an equivalence of categories, i.e. $\mathcal{A}$ and $e\mathcal{A}e$ are Morita equivalent.

An indecomposable direct summand of $\mathcal{A}_\mathfrak{a}$ is called a PIM.

(A PIM in the sense of Lecture 2 is the Brauer character of a PIM of $FG$, extended by 0 from $G_{p'}$ to $G$.)

A projective $\mathcal{A}$-module is a direct sum of PIMs.

A finite-dimensional $F$-algebra $\mathcal{B}$ is Morita equivalent to $\mathcal{A}$, if $\mathcal{B} \cong \text{End}_{\mathcal{A}}(Q)$ for a projective module $Q$ of $\mathcal{A}$ containing every PIM of $\mathcal{A}$ (up to isomorphism) as a direct summand.

Morita equivalent algebras have “the same” representations.
Let $H \leq G$ with $\text{char}(F) \nmid |H|$. Then

$$e := e_H := \frac{1}{|H|} \sum_{x \in H} x \in FG$$

is a suitable idempotent.

Other idempotents can be used, e.g.,

$$e = \frac{1}{|H|} \sum_{x \in H} \lambda(x^{-1})x \in FG,$$

where $\lambda : H \rightarrow F^*$ is a homomorphism (Noeske, 2005).
Let $e := e_H = 1/|H| \sum_{x \in H} x$ be as above.

Let $V$ be the permutation $FG$-module w.r.t. an action of $G$ on the finite set $\Omega$. Then $Ve$ is the set of $H$-fixed points in $V$.

**Task:** Given $g \in G$, determine the action of $ege$ on $Ve$, without the explicit computation of the action of $g$ on $V$.

**Theorem (Thackray and Parker, 1981)**

*This can be done!*
Let $\Omega_1, \ldots, \Omega_m$ be the $H$-orbits on $\Omega$.

The orbits sums $\hat{\Omega}_j := \sum_{\omega \in \Omega_j} \omega \in V$ form a basis of $Ve$.

W.r.t. this basis, the $(i, j)$-entry $a_{ij}$ of the matrix of $ege$ on $Ve$ equals

$$a_{ij} = \frac{1}{|\Omega_j|} |\Omega_ig \cap \Omega_j|.$$
To perform these computations, we need to be able to

1. compute \( \Omega_j \) and \( |\Omega_j| \), \( 1 \leq j \leq m \),
2. decide \( \omega \in \Omega_j \) for given \( \omega \in \Omega \) and \( 1 \leq j \leq m \).

In actual applications, \( |\Omega| \approx 10^{15} \), so the elements of \( \Omega \) cannot be stored in memory.

Parker and Wilson suggested Direct Condensation methods; these were later extended and implemented by Cooperman, Lübeck, Müller and Neunhöffer.

**Principal idea:** Enumerate the \( H \)-orbits \( \Omega_j \) by suborbits of subgroups \( U \leq H \). Iterate this idea.

Details depend on the realisation of the action on \( \Omega \).
Let $V$ and $W$ be two $FG$-modules.

**Task:** Given $g \in G$, determine the action of $e g e$ on $(V \otimes W)e$, without the explicit computation of the action of $g$ on $V \otimes W$.

**Theorem (Lux and Wiegelmann, 1997)**

*This can be done!*

Let $M$ be a subgroup of $G$ and let $W$ be an $FM$-module.

The **induced module** is the $FG$-module $W \otimes_{FM} FG$.

**Task:** Given $g \in G$, determine action of $e g e$ on $(W \otimes_{FM} FG)e$, without the explicit computation of the action of $g$ on $W \otimes_{FM} FG$.

**Theorem (Müller and Rosenboom, 1997)**

*This can be done!*
**HOMOMORPHISM SPACES**


Then $\text{Hom}_{FM}(V, W)$ is a right $FG$-module:

$$v(\varphi g) := (gv)\varphi, \quad v \in V, \varphi \in \text{Hom}_{FM}(V, W), g \in G.$$ 

**EXAMPLES**

1. $\text{Hom}_{FM}(FG, F) \cong \text{permutation module corresponding to permutation action of } G \text{ on } \Omega := M\backslash G.$

2. $\text{Hom}_F(V^*, W) \cong V \otimes W \text{ for } V, W \in \text{mod-}FG.$
   $(V^* = \text{Hom}_F(V, F).)$

3. $\text{Hom}_{FM}(FG, W) \cong W \otimes_{FM} FG.$

Lux, Neunhöffer, Noeske develop general Condensation programs for such homomorphism spaces.
Condensation: Some Applications

Benson, Conway, Parker, Thackray, Thompson, 1980:
Existence of $J_4$.

Thackray, 1981:
2-modular character table of McL.
Answer to a question of Brauer.

Cooperman, H., Lux, Müller, 1997:
Brauer tree of Th modulo 19.
dim($V$) = 976 841 775, dim($V_e$) = 1403.

Müller, Neunhöffer, Röhr, Wilson, 2002:
Brauer trees of Ly modulo 37 and 67.
dim($V$) = 1 113 229 656.

More applications later.
THE BASIC ALGEBRA

DEFINITION

A finite-dimensional $F$-algebra $\mathcal{B}$ is called basic, if

$$\mathcal{B} = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$$

with PIMs $Q_i$ such that $Q_i \not\cong Q_j$ for $1 \leq i \neq j \leq n$.

Alternatively, if $\mathcal{B}/J(\mathcal{B})$ is a direct sum of division algebras.

FACTS

Let $P_1, \ldots, P_n$ be the PIMs of $\mathbb{A}$ (up to isomorphism). Then $\mathcal{B} := \text{End}_\mathbb{A}(P_1 \oplus \cdots \oplus P_n)$ is a basic algebra Morita equivalent to $\mathbb{A}$, the basic algebra of $\mathbb{A}$.

This is the smallest algebra Morita equivalent to $\mathbb{A}$.
Condensation
An Example: The Fischer Group $Fi_{23}$ Modulo 2

Morita Equivalent Algebras

If $\dim(A)$ is large, it may be too difficult to construct the basic algebra of $A$ explicitly.

Klaus Lux uses Condensation to construct algebras of feasible dimensions, Morita equivalent to (blocks of) group algebras $FG$.

Need idempotent $e \in FG$ with $Se \neq 0$ for all simple $FG$-modules $S$ (or all simple modules in a block).

This can be checked with the modular character table of $G$, if $e = e_H$ for some $H \leq G$ with $\text{char}(F) \nmid |H|$.

Example: Principal block $B_0$ of $HS$ modulo 5, $|H| = 192$.
$\dim(B_0) = 15\,364\,500$, $\dim(e_H B_0 e_H) = 767$.

See Klaus Lux, Faithful Condensation for Sporadic Groups, (http://math.arizona.edu/~klux/habil.html)

Applications: Cartan matrices for group algebras, cohomology computations
Condensation: History

H \leq G

H x H in F_6

Multiplication in F_6

Ponder double cases H x H

New multiplication

H x H, H x H = H x H

\sigma_H = \text{image of } \sigma(\sum h)

\sigma_H(x H x y H) = \sigma(H x H)

Use this line to define x.
We investigate $Ve$ through the MeatAxe, using matrices of generators of $eFGe$.

**Question (The Generation Problem)**

*How can $eFGe$ be generated with “a few” elements?*

If $\mathcal{E} \subseteq FG$ with $F\langle \mathcal{E} \rangle = FG$, then in general $F\langle e\mathcal{E}e \rangle \subseteq eFGe$.

- Let $\mathcal{C} := F\langle e\mathcal{E}e \rangle \leq eFGe$.
  Instead of $Ve$ we consider the $\mathcal{C}$-module $Ve|_\mathcal{C}$.

- We can draw conclusions on $V$ from $Ve$, but *not* from $Ve|_\mathcal{C}$.
**Theorem (F. Noeske, 2005)**

Let $H \trianglelefteq N \leq G$. If $\mathcal{T}$ is a set of double coset representatives of $N \setminus G/N$ and $\mathcal{N}$ a set of generators of $N$, then we have for $e = e_H$:

$$eFGe = F\langle e\mathcal{N}e, e\mathcal{T}e \rangle$$

as $F$-algebras.

More sophisticated results by Noeske on generation are available, but have not found applications yet.

**Matching Problem:** Let $e, e' \in FG$ be idempotents. Suppose $S, S' \in \text{mod-}FG$ are simple, and we know $Se$ and $S'e'$. Can we decide if $S \cong S'$? Yes! (Noeske, 2008)
Condensing Projective Modules

Not a new idea, but now feasible through
- improved Condensation techniques
- programs by Jon Carlson for matrix algebras (see next lecture)

If \( P = eFG \) is projective, then \( \text{End}_{FG}(P) = eFGe = Pe \) decomposes in the same way as \( P \).

Example \((G = Th, \ p = 5)\)

1. \((Done \ in \ 2007 \ with \ Jon \ Carlson)\):
   \[ P = e_HFG, \ for \ H = 3xG_2(3), \ \dim(P) = 7,124,544,000, \]
   \[ \dim_F(\text{End}_{FG}(P)) = 788 \leadsto \text{some progress} \]

2. \((Envisaged)\):
   \[ \dim(Q) = 43,957,879,875, \ \dim_F(\text{End}_{FG}(Q)) = 21,530 \]
   \[ \leadsto \text{almost finish Th modulo 5} \]
Let $G$ denote the Fischer group $Fi_{23}$. This is a sporadic simple group of order

$$4,089,470,473,293,004,800.$$ 

$G$ has a maximal subgroup $M$ of index 31,671, isomorphic to $2.Fi_{22}$, the double cover of the Fischer group $Fi_{22}$. 

In joint work with Max Neunhöffer and F. Noeske we have computed the 2-modular character table of $G$. 

**SOME REPRESENTATIONS OF $Fi_{23}$**

In the following, let $F$ denote a finite field of characteristic 2.

Let $\Omega := M \setminus G$ and let $V$ denote the corresponding permutation module over $F$ (thus $\dim_F(V) = 31\,671$).

Using the MeatAxe we found: $V$ contains composition factors $1, 782, 1\,494, 3\,588, 19\,940$ (denoted by their degrees).

(This took about 4 days of CPU time in 8 GB main memory.)

Using the Condensation we analysed the ten tensor products:

$$782 \otimes 782, 782 \otimes 1\,494, \ldots, 19\,940 \otimes 19\,940.$$ 

**Note:** $\dim_F(19\,940 \otimes 19\,940) = 367\,603\,600$.

One such matrix over $\mathbb{F}_2$ would need $\approx 18\,403\,938$ GB.
The Condensation for $Fi_{23}$

1. We took $H \leq G$, $|H| = 3^9 = 19\,683$.

2. We found that $eFGe$ and $FG$ are Morita equivalent (a posteriori).

3. $\dim_F (19\,940 \otimes 19\,940)e = 25\,542$.

   One such matrix over $\mathbb{F}_2$ needs $\approx 77.8$ MB.

   About 1 week of CPU time to compute the action of one element $ege$ on $(19\,940 \otimes 19\,940)e$.

4. Every irreducible $FG$-module (of the principal 2-block) occurs in $19\,940 \otimes 19\,940$. 
The results of the Condensation and further computations with Brauer characters using GAP and MOC gave all the irreducible 2-modular characters of $G$.

Degrees of the irreducible 2-modular characters of $Fi_{23}$:

\begin{align*}
1, & \quad 782, & \quad 1494, & \quad 3588, \\
19940, & \quad 57408, & \quad 79442, & \quad 94588, \\
94588, & \quad 583440, & \quad 724776, & \quad 979132, \\
1951872, & \quad 1997872, & \quad 1997872, & \quad 5812860, \\
7821240, & \quad 8280208, & \quad 17276520, & \quad 34744192, \\
73531392, & \quad 97976320, & \quad 166559744, & \quad 504627200, \\
504627200. & & & 
\end{align*}
REFERENCES


Thank you for your attention!