Distinguished representations of finite classical groups

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Congratulations

Happy Birthday to You, Professor Zalesski!
Contents

1. The Steinberg representation
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These are the classical groups defined over finite fields. i.e., linear groups preserving a form of degree 2.

**Examples**

- $\text{GL}_n(q)$, $\text{GU}_n(q)$, $\text{Sp}_{2n}(q)$, $\text{SO}_{2n+1}(q)$ ... ($q$ a prime power)
- E.g., $\text{Sp}_{2n}(q) = \{ A \in \text{GL}_{2n}(q) \mid A^{tr} \tilde{J} A = \tilde{J} \}$, where
  \[ \tilde{J} = \begin{bmatrix} J & \cdot & \cdot & \cdot \\ -J & 1 \end{bmatrix} \quad \text{with} \quad J = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ & 1 & \cdot & \cdot \\ & & \ddots & \cdot \\ & & & 1 \end{bmatrix} \in \mathbb{F}_{q}^{n \times n}. \]
- Related groups, e.g., $\text{SL}_n(q)$, $\text{PSL}_n(q)$, $\text{CSp}(q)$ etc. are also classical groups.
Distinguished Representations

1. Representations in this talk are taken mostly over the field \( \mathbb{C} \) of complex numbers.

2. Distinguished representations are families of representations obtained by a uniform construction.

3. Some of the distinguished representations also exist for other groups of Lie type, or for infinite classical groups, others exist only for some of the classical groups.
Frobenius, 1896: \( \text{PSL}_2(p) \) has irreducible representation of degree \( p \).

Schur, 1907: \( \text{SL}_2(q) \) has irreducible representation of degree \( q \).

Steinberg, 1951: \( \text{GL}_n(q) \) has irreducible representation of degree \( q^{n(n-1)/2} \).

Steinberg, 1956: Finite classical groups of characteristic \( p \) have an irreducible representation whose degree equals the order of a Sylow \( p \)-subgroup.

Steinberg, 1957: Same result as above for groups with split \( BN \)-pairs of characteristic \( p \).
Steinberg, *A geometric approach to the representations of the full linear group over a Galois field*. TAMS 1951.

Steinberg constructs all *unipotent* representations of $\text{GL}_n(q)$. These are in bijection with the partitions of $n$.

In particular, the representation

$$
\Gamma(1^n) = \sum_{\nu} \varepsilon(\nu) \text{Ind}_{P_\nu}^G(\mathbb{C}).
$$

Here, $\nu$ runs through the compositions of $n$, $\varepsilon(\nu)$ is a sign, and $\text{Ind}_{P_\nu}^G(\mathbb{C})$ is the permutation module on the flags of shape $\nu$.

Steinberg shows that $\Gamma(1^n)$ has degree $q^{n(n-1)/2}$. 
Construction of the Steinberg repr’n, II


Steinberg constructs the Steinberg representation for all finite groups $G$ with a split $BN$-pair of characteristic $p$:

Let $W$ be the Weyl group, $U \leq B$ a Sylow subgroup of $G$.

Put

$$e := \left( \sum_{b \in B} b \right) \left( \sum_{w \in W} (-1)^{\ell(w)} w \right).$$

Then $e \mathbb{C} G$ has basis \{eu | u \in U\} and affords the Steinberg representation $St_G$. 
Construction of the Steinberg repr’n, III

Charles W. Curtis, 1965: $G$ a finite group with $BN$-pair with Coxeter system $(\mathcal{W}, S)$.

The character $\text{ch}(\text{St}_G)$ of the Steinberg representations equals

$$\text{ch}(\text{St}_G) = \sum_{J \subset S} (-1)^{|J|} \text{Ind}_{P_J}^G(1),$$

where $P_J$ is the standard parabolic subgroup corresponding to $J$, and $1$ denotes the trivial character.

$\text{St}_G$ is the unique constituent of $\text{Ind}_B^G(\mathbb{C})$ which is not a component of $\text{Ind}_{P_J}^G(\mathbb{C})$ for $J \neq \emptyset$ [$B = P_\emptyset$].
L. Solomon 1969, Curtis, Lehrer, Tits, 1979:

Let $G$ be a finite group with $BN$-pair of rank $r$ and let $\Delta$ be the Tits building of $G$. Then

$$H_{r-1}(\Delta) \otimes_{\mathbb{Z}} \mathbb{C}$$

affords the Steinberg representation of $G$. 
Let $G$ be a finite group of Lie type, $U$ its unipotent subgroup.

$U = \prod_{\alpha \in \Phi^+} U_{\alpha}$, where $\Phi^+$ is the set of positive roots and $U_{\alpha}$ a root subgroup.

$\lambda : U \to \mathbb{C}^*$ is in general position, if $\lambda_{\downarrow U_{\alpha}} \neq 1$ if $\alpha$ is simple, and $\lambda_{\downarrow U_{\alpha}} = 1$, otherwise.

$$\Gamma := \text{Ind}_U^G(\lambda),$$

where $\lambda$ is a linear representation of $U$ in general position, is called a Gelfand-Graev representation of $G$.

In particular, the degree of $\Gamma$ equals $|G|/|U| = |G|/\deg(\text{St}_G)$.

For some groups, e.g., $\text{Sp}_{2n}(q)$, $q$ odd, there is more than one Gelfand-Graev representation.
Some properties of the Gelfand-Graev repr’n

Theorem (Gelfand-Graev ’62, Yokonuma ’67, Steinberg ’68)

A Gelfand-Graev representation $\Gamma$ is multiplicity free.

Idea of prove: $\text{End}_{CG}(\Gamma)$ is abelian.

The characters of the irreducible constituents of $\Gamma$ are called regular characters of $G$. E.g., $\text{ch}(\text{St}_G)$ is regular.

Note:

$$\text{St}_G \otimes_{\mathbb{C}} \Gamma_G$$

equals the regular representation of $G$. 
History of the Weil Representation

1961: Bolt, Room & Wall define Weil representations for finite conformal symplectic groups.
1964: Weil constructs Weil representations associated to symplectic vector spaces over local fields.
1972: Ward defines Weil representations for finite symplectic groups.

Howe (1973), Isaacs (1973), Lehrer (1974), Seitz (1975) & Gérardin (1975) contribute to different aspects of the
- construction of the Weil representations for finite general linear and unitary groups,
- determination of the characters of the Weil representations,
- determination of the irreducible constituents of the Weil representations.
Weil character for general unitary groups

$\varepsilon = \pm 1$, $G = \text{GL}_n(\varepsilon q)$ acting on $V$ \hspace{1cm} $[\text{GL}_n(-q) := \text{GU}_n(q)]$

$\omega$ character of Weil representation of $G$

$\omega(1) = q^n$

$\omega(g) = \varepsilon^n (\varepsilon q)^{N(V, g)}$, $g \in G$ \hspace{1cm} $[N(V, g) = \dim(\ker(1 - g))]$

For $G = \text{GL}_n(q)$, $\omega$ is the permutation character on $V$.

$\omega$ is multiplicative, i.e.,

if $V = V_1 \oplus V_2$, then $\omega_{G(V_1) \times G(V_2)} = \omega_{G(V_1)} \boxtimes \omega_{G(V_2)}$. 
Construction of Weil representation, I

Suppose $G = \text{GU}_n(q)$ acting on $V = \mathbb{F}_{q^2}^n$, preserving the Hermitian form $\beta$.

For $v \in V$, let $v^*$ denote the linear form determined by $\beta$ and $v$.

The Heisenberg group of $\beta$ is the group

$$H = \left\{ \begin{bmatrix} 1 & v^* & z \\ 0 & I_n & v \\ 0 & 0 & 1 \end{bmatrix} \mid v \in V, z \in \mathbb{F}_{q^2}, z + v^*(v) + z^q = 0 \right\}.$$

This is a special $p$-group of order $q^{2n+1}$.

The centre of $H$ is elementary abelian of order $q$. 
Let $\zeta$ be an irreducible representation of $H$ with $Z(H)$ not in kernel. ($H$ has $q - 1$ distinct such representations.)

$G$ acts on $H$ in a natural way, fixing $Z(H)$ element-wise.

Consider the semidirect product $P := HG$.
Then $\zeta$ is invariant in $P$.

**Fact:** $\zeta$ extends to an irreducible representation $\hat{\zeta}$ of $P$.

$\Omega_n := \text{Res}_G^P(\hat{\zeta})$ is the Weil representation of $G = GU_n(q)$.

Each such $\zeta$ yields the same $\Omega_n$.

An analogous construction works for the symplectic groups.
$G = G_n(q) \in \{GL_n(q), Sp_{2n}(q) (q \text{ odd}), GU_{2n}(q), GU_{2n+1}(q)\}$

$\text{St}_n$: Steinberg representation of $G$

$\Omega_n$: Weil representation of $G$

$\Gamma_n$: Gelfand-Graev representation of $G$

$WSt_n := \Omega_n \otimes \text{St}_n$: Weil-Steinberg representation of $G$

[Zalesski, H., 2008]
Weil-Steinberg representation, I

\[ G = \text{GL}_n(q), \text{ } V \text{ natural vector space for } G \]

\[ P_m: \text{stabilizer of } m\text{-dim. subspace of } V \text{ } (0 \leq m \leq n) \]

\[ L_m = \text{GL}_m(q) \times \text{GL}_{n-m}(q) \text{ Levi subgroup of } P_m \]

**Theorem (Zalesski, H., '08)**

\[ \Omega_n \otimes \text{St}_n = \sum_{m=0}^{n} \text{Ind}_{P_m}^{G} \left( \text{Infl}_{L_m}^{P_m} (\text{St}_m \boxtimes \Gamma_{n-m}) \right). \]

Brundan, Dipper, Kleshchev (2001), Memoirs, Section 5:
Contains related and more general results.
THE STEINBERG REPRESENTATION
THE GELFAND-GRAEV REPRESENTATIONS
THE WEIL REPRESENTATION
THE WEIL-STEINBERG REPRESENTATION

**Weil-Steinberg Representation, II**

$G$ as above, $G \neq \text{GL}_n(q)$, $V$ natural vector space for $G$

$P_m$: stabilizer of $m$-dim. isotropic subspace of $V$ ($0 \leq m \leq n$)

$L_m = \text{GL}_m(q') \times G_{n-m}(q)$ Levi subgroup of $P_m$

[$q' = q^2$ if $G$ is unitary, $q' = q$, otherwise]

**Theorem (Zalesski, H., ’08)**

$$\Omega_n \otimes \text{St}_n = \sum_{m=0}^{n} \text{Ind}_{P_m}^{G} \left( \text{Infl}_{L_m}^{P_m} (\text{St}_m \boxtimes \Gamma'_{n-m}) \right).$$

$\Gamma'_{n-m}$: truncated Gelfand-Graev representation of $G$, defined via Deligne-Lusztig theory
Suppose $G = GU_n(q)$.

$$\text{Irr}(G) = \bigcup_{s \in S} \mathcal{E}(G, s),$$

a disjoint union of **Lusztig series** $\mathcal{E}(G, s)$. Here, $\mathcal{S}$ is a set of representatives of the conjugacy classes of semisimple elements of $G$. Every $\mathcal{E}(G, s)$ contains exactly one regular character, $\chi_s$, and

$$\text{ch}(\Gamma_G) = \sum_{s \in \mathcal{S}} \chi_s.$$

Now $\text{ch}(\Gamma'_s)$ is the sum of those $\chi_s$ such that $s$ has no eigenvalue $-1$ on $V$. [1 if $q$ is even.]

A similar definition applies for the symplectic groups.
Suppose $G \neq \text{GL}_n(q)$.

Write $\omega$ and $\text{st}$ for the characters of $\Omega_n$ and $\text{St}_n$, respectively.

1. \[
\omega \cdot \text{st} = \sum_{(T, \theta)} \frac{\varepsilon_g \varepsilon_T(\omega \cdot \text{st}, R_{T, \theta})}{|W(T, \theta)|} R_{T, \theta}
\]

2. \[(\omega \cdot \text{st}, R_{T, \theta}) = (\omega, \text{st} \cdot R_{T, \theta}) = (\omega, \text{Ind}_T^G(\theta)) = (\text{Res}_T^G(\omega), \theta)\]

3. Use multiplicativity of $\omega$ to compute $\text{Res}_T^G(\omega)$.

4. If $(T, \theta)$ corresponds to a semisimple element without eigenvalue $-1$ on $V^*$, then $(\text{Res}_T^G(\omega), \theta) = 1$.

5. If $(\text{Res}_T^G(\omega), \theta) \neq 0$, then $T$ stabilizes an isotropic subspace of $V$. 
Consequences, I

Corollary 1: $G \neq \text{GL}_n(q)$. Then $\Omega_n \otimes \text{St}_n$ is multiplicity free.

This is not the case for $G = \text{GL}_n(q)$.

Corollary 2 (Schröer): If $G = \text{GL}_n(q)$ then

$$
\Omega_n \otimes \text{St}_n = \sum_{m=0}^{n} \sum_{s' \in \mathcal{S}_m'} \sum_{k=0}^{\lfloor (n-m)/2 \rfloor} (n - m - 2k + 1) \chi_{(n-m-k,k)';s'}.
$$

Here, $\mathcal{S}_m'$ is a set of representatives of the conjugacy classes of semisimple elements of $\text{GL}_m(q)$ without eigenvalue 1, and

$\chi_{(n-m-k,k)';s'} \in \text{Irr}(G)$ has Jordan decomposition $(s, \psi)$ with

$s = 1_{n-m} \times s'$, $C_G(s) = \text{GL}_{n-m}(q) \times C_{\text{GL}_m(q)}(s')$,

$\psi = \chi_{(n-m-k,k)'} \boxtimes \text{St} \in \text{Irr}_u(C_G(s))$. 
Consequences, II

\[ G = G_n(q) \in \{ \text{GL}_n(q), \text{Sp}_{2n}(q) (q \text{ odd}), \text{GU}_{2n}(q), \text{GU}_{2n+1}(q) \} \]

\( V \) natural vector space for \( G \), \( v \in V \)

Corollary (An, Brunat, Zalesski, H., ’06 – ’08)

Let \( H \) be the stabilizer of \( v \) in \( G \). Then

\( \text{Res}_H^G(\text{St}_G) \) is multiplicity free.

Question

Is this also true for \( G = \text{SO}_{2n+1}(q), \text{SO}_{2n}^\pm(q) \)?
\[ G = \text{Sp}_{2n}(q), \ q \text{ even.} \]

No Weil representation is defined for this group.

Replace \( \Omega_n \) by \textit{generalized spinor representation} \( \Sigma_n \) with highest weight \((0, \ldots, 0, q - 1)\), \( \deg(\Sigma_n) = q^n \).

View \( \text{St}_n \) as a projective, irreducible \( \overline{\mathbb{F}}_q \)-representation of \( G \). Then \( \Sigma_n \otimes \text{St}_n \) is a projective \( \overline{\mathbb{F}}_q \)-representation of \( G \).

By general theory, \( \Sigma_n \otimes \text{St}_n \) lifts to an ordinary representation, which we call the \textbf{Weil-Steinberg representation} \( \text{WSt}_n \) of \( G \).
The same formula as for odd $q$ holds:

**Theorem (Zalesski, H., ’08)**

$$WSt_n = \sum_{m=0}^{n} \text{Ind}_P^G \left( \text{Infl}_{L_m}^P (\text{St}_m \boxtimes \Gamma'_{n-m}) \right).$$

We also find the decomposition of $WSt_n$ into PIMs.

**Theorem (Zalesski, H., ’08)**

$$WSt_n = \sum_{j=0}^{q-1} \Phi_{\nu_j},$$

where $\Phi_{\nu_j}$ is the (lift of) the projective cover of the simple representation $\nu_j$ with highest weight $(q-1, \ldots, q-1, j)$. 
Let $G$ be a connected reductive group defined over $\mathbb{F}_p$. Let $q$ be a power of $p$. Then $G := G(q)$ is a finite group of Lie type.

Let $M$ be a simple $\mathbb{F}_q G$-module.

Call $M$ $G$-regular, if for every rational maximal torus $T$ of $G$, distinct $T$-weight spaces of $M$ are non-isomorphic $T(q)$-mod's.

**Theorem (Zalesski, H., ’09)**

Suppose that $M$ is $G$-regular and $d = \dim(M)$. Let $\varphi$ denote the Brauer character of $\text{Res}^G_M(G)$. Then $\varphi \otimes \text{ch}(\text{St}_G)$ is a sum of $d$ distinct regular characters of $G$.

**Theorem (Zalesski, H., ’09)**

Given $G$ and $M$, there are only finitely many $q$ such that $M$ is not $G(q)$-regular.
Thank you for your attention!