MORPHISMS AND INVERSE PROBLEMS FOR DARBOUX INTEGRATING FACTORS

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ABSTRACT. Polynomial vector fields which admit a prescribed Darboux integrating factor are quite well-understood when the geometry of the underlying curve is nondegenerate. In the general setting morphisms of the affine plane may remove degeneracies of the curve, and thus allow more structural insight. In the present paper we establish some properties of integrating factors subjected to morphisms, and we discuss in detail one particular class of morphisms related to finite reflection groups. The results indicate that degeneracies for the underlying curve generally impose additional restrictions on vector fields admitting a given integrating factor.

1. INTRODUCTION AND PRELIMINARIES

This paper continues, and to some extent concludes, our work on inverse problems in the Darboux theory of integrability in the affine plane. The inverse problem for Darboux integrating factors is to characterize and determine all polynomial vector fields which admit a prescribed integrating factor. The inverse problem for invariant algebraic curves (which is a part of the former) asks for all polynomial vector fields which admit a given collection of algebraic curves as invariant sets.

Inverse problems in the Darboux theory of integrability are of interest because their solution is necessary to identify and classify the vector fields admitting a Darboux integrating factor. Moreover, such inverse problems have useful applications. For instance, Christopher [1] used an inverse problem to produce polynomial vector fields with algebraic limit cycles, and in [6] solutions of inverse problems were employed to determine vector fields of small degree with a prescribed limit cycle configuration.

In a nondegenerate geometric setting both inverse problems were essentially resolved in [2] and [3]; see also earlier work in [10], [5], [7]. Moreover, the inverse problem for curves is generally well-understood and algorithmically accessible; see [2].

The main result of [3] states that the linear space of vector fields admitting a given integrating factor is finite dimensional modulo a subspace of "trivial" vector fields, provided the underlying geometry is nondegenerate. This finiteness result was extended to arbitrary geometry in [4] with the help of sigma processes, a particular class of morphisms of the affine plane.

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In the present paper we will consider general morphisms of the affine plane and discuss the behavior of invariant curves and integrating factors subject to morphisms.

Assume that irreducible, pairwise relatively prime polynomials $f_1, \dots, f_r \in \mathbb{C}[x,y]$ are given, and abbreviate $f = f_1 \cdots f_r$. We will consider complex polynomial vector fields

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

sometimes briefly written as $X = (P,Q)^t$. Our principal focus will be on the inverse problem for Darboux integrating factors. Thus fix nonzero complex constants $d_1, \dots, d_r$, and consider all vector fields admitting the Darboux integrating factor

$$\left(f_1^{d_1} \cdots f_r^{d_r}\right)^{-1}. \tag{1}$$

It is known (cf. [7], [2], [10]) that for any such vector field the complex zero set of f is invariant; equivalently there is a polynomial L (called the cofactor of f) such that

$$Xf = L \cdot f. \tag{2}$$

The respective zero sets of f and f_i in $\mathbb{C}^2$ will be denoted by C and C_i. As usual, we call a point z with $f(z) = f_x(z) = f_y(z) = 0$ a singular point of C, and similarly for the C_i. The *Hamiltonian vector field* of f is defined by

$$X_f = -f_y\frac{\partial}{\partial x} + f_x\frac{\partial}{\partial y}.$$

Following and slightly modifying [7], two generic nondegeneracy conditions were introduced in [3]:

(ND1) Each C_i is nonsingular.
(ND2) All singular points of C have multiplicity one.

Thus when two irreducible components of C intersect, they intersect transversely, and no more than two irreducible components intersect at any point.

The vector fields admitting the integrating factor (1) form a linear space $\mathcal{F}$. We first exhibit some of its elements; cf. [10] and [3]. Given an arbitrary polynomial g, define

$$Z_g = Z_g^{(d_1,\dots,d_r)} \tag{3}$$

to be the Hamiltonian vector field of $g/\left(f_1^{d_1-1} \cdots f_r^{d_r-1}\right)$. Then

$$f_1^{d_1} \cdots f_r^{d_r} \cdot Z_g^{(d_1,\dots,d_r)} = f X_g - \sum_{i=1}^{r} (d_i - 1) g \frac{f}{f_i} X_{f_i} \in \mathcal{F} \tag{4}$$

is easily verified. Note that the last expression indeed defines a polynomial vector field, and that the property of admitting the integrating factor (1) does not depend on the irreducibility or the relative primeness of the f_i. The vector fields of this particular type will be called *trivial*. They form a subspace $\mathcal{F}^0$ of $\mathcal{F}$. In presence of the geometric nondegeneracy conditions (ND1) and (ND2) we showed in [3] that the codimension of $\mathcal{F}^0$ in $\mathcal{F}$ is finite. With the help of sigma processes this result was extended to the general case in [4]. Nontrivial vector fields may exist; for instance in the case $d_1 = \cdots = d_r = 1$ all vector fields of the form

$$X = \sum \alpha_i \frac{f}{f_i} X_{f_i} + f \cdot X_h \tag{5}$$

with constants α_i and an arbitrary polynomial h admit the integrating factor f^{-1}. For nondegenerate geometry one knows that there are no others; see [3].

Generally, it seems desirable to obtain better insight into the structure of vector fields with a given integrating factor when the underlying curves have degenerate singular points. As will be shown in this article, morphims can be useful for this purpose. In Section 2 we record some basic technical results and properties. In Section 3 we discuss a particular class of morphisms related to reflection groups in the plane, and as an application we obtain a class of examples with degenerate irreducible underlying curves. These examples show that degenerate geometry may impose obstructions to the existence of nontrivial vector fields with prescribed Darboux integrating factor. Indeed, after a transformation to remove degeneracies, the quotient space $\mathcal{F}/\mathcal{F}^0$ will generally have higher dimension than in the degenerate setting.

2. Morphisms

Morphisms of the affine plane may transform certain degenerate curve configurations into nondegenerate ones (in the sense that (ND1) and (ND2) hold) or at least into less degenerate settings. This observation opens an approach to solutions of inverse problems with degenerate geometry. Sigma processes were employed in Section 5 of [2] for the inverse problem for curves, and in Section 4 of [3] to investigate the inverse problem for integrating factors. Here we discuss general morphisms of the affine plane and the behavior of invariant curves and integrating factors subject to morphisms.

Consider a polynomial map

$$\Phi : \mathbb{C}^2 \to \mathbb{C}^2, \quad \det D\Phi \neq 0, \tag{6}$$

thus the image of Φ is dense in the plane (see e.g. Shafarevich [8]) and we have local analytic invertibility on an open and dense set. The comorphism of Φ assigns to every polynomial $g \in \mathbb{C}[x, y]$ the polynomial

$$\widehat{g} := g \circ \Phi, \tag{7}$$

and to every polynomial vector field $X = P\,\partial/\partial x + Q\,\partial/\partial y$ the rational vector field

$$\Phi_*(X) = D\Phi(x, y)^{-1} \begin{pmatrix} P\,(\Phi(x, y)) \\ Q\,(\Phi(x, y)) \end{pmatrix} \tag{8}$$

as well as the polynomial vector field

$$\widehat{X} = \det(D\Phi(x, y)) \cdot \Phi_*(X). \tag{9}$$

Note that these definitions also are applicable to analytic functions and vector fields.

Proposition 1. *Let $g = g_1 \cdots g_r$ be a polynomial, with irreducible factors g_i, and X a polynomial vector field on $\mathbb{C}^2$.*

(a) *The zero set of g is invariant for X if and only if the zero set of $\hat{g}$ is invariant for $\widehat{X}$:*

$$X(g) = K \cdot g \Leftrightarrow \widehat{X}(\hat{g}) = \widehat{K} \cdot \hat{g} \quad \text{with } \widehat{K} := \det D\Phi \cdot (K \circ \Phi).$$

(b) *Given constants $d_1, \ldots, d_r$, the vector field X admits the integrating factor $g_1^{-d_1} \cdots g_r^{-d_r}$ if and only if the vector field $\widehat{X}$ admits the integrating factor $\hat{g}_1^{-d_1} \cdots \hat{g}_r^{-d_r} = (g_1^{-d_1} \cdots g_r^{-d_r}) \circ \Phi$.*

Proof. Part (a) is a direct consequence of the identity

$$\Phi_*(X)\,(g\circ\Phi) = (X(g))\circ\Phi.$$

For part (b) see e.g. [10], Corollary 1.3. □

Some elementary properties of the transformations will be collected next. Proofs are included for the sake of completeness.

Lemma 2. *Let Φ be as in (6).*

(a) *For any analytic f one has the identity*

$$\widehat{X_f} = X_{\widehat{f}}.$$

(b) *Given polynomials $f_1,\ldots,f_r$ and g, and the vector field defined in (3), one has the identity*

$$\hat{f}_1^{d_1}\cdots\hat{f}_r^{d_r}Z_{\widehat{g}} = \widehat{f_1^{d_1}\cdots f_r^{d_r}Z_g}.$$

Proof. Note that

$$X_f(x,y) = J\cdot Df(x,y)^t, \quad \text{with } J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $.^t$ denotes transposition. Using the identity

$$J\cdot A^t = A^*\cdot J$$

for arbitrary matrices A, with A^* the adjoint of A, and

$$D\hat{f}(x,y)^t = D\Phi(x,y)^t\cdot Df(x,y)^t,$$

we find

$$X_{\hat{f}}(x,y) = \det D\Phi(x,y)\,D\Phi(x,y)^{-1}\cdot J\cdot Df(\Phi(x,y))^t$$

as asserted in part (a). Part (b) is now immediate from the definitions. □

Remark 3. (a) Proposition 1 and Lemma 2 in particular apply to automorphisms of the affine plane. Mutatis mutandis, results about vector fields that admit invariant curves or Darboux integrating factors are unaffected by automorphisms. This fact has been often tacitly used, such as in [3]. But automorphisms will not remove or create degeneracies, and therefore are of less interest in the present paper.

(b) To apply Proposition 1 and Lemma 2, one may start with a morphism Φ that turns a polynomial f, with degenerate geometry of the underlying curve C, to a polynomial $\widehat{f}$ with nondegenerate, or less degenerate, geometry of the underlying curve. For the transformed polynomial it may be possible to determine all vector fields that admit $\widehat{f}$, resp. a particular Darboux integrating factor. There remains the problem to decide under what circumstances such a vector field Y is of the type $\widehat{X}$ as given in (9). Generally this decision poses a nontrivial problem, but in the next section we consider a class of morphisms for which it is easily manageable.

(c) In Lemma 2(b), some $\hat{f}_i$ may be reducible even if the f_i are irreducible. Note that the expression defining $\hat{f}_1^{d_1}\cdots\hat{f}_r^{d_r}Z_{\widehat{g}}$ will change its appearance when it is rewritten in the form (4) with irreducible factors.

To mention a substantial application of morphisms, in [4] sigma processes (and the Tarski-Seidenberg theorem) were used to extend the main result of [3] from nondegenerate geometry to the general case and prove finite dimension of $\mathcal{F}/\mathcal{F}^0$. The critical argument in [4] is concerned with the behavior of trivial vector fields under transformation. On the other hand, for sigma processes it seems hard to understand the behavior of nontrivial vector fields under transformation, and they do not seem well-suited for computations in actual examples. As a counterpoint, we will therefore present a class of morphisms which pose few computational problems, and provide easy access to relevant applications.

3. Invariants of reflection groups

In this section we discuss morphisms related to reflection groups, which allow the investigation of certain complicated geometric settings with little computational effort. In particular, the problem to decide whether $Y = \widehat{X}$ for some X (see Remark 3(b)) is easily accessible.

Recall that a reflection of the plane (in the sense of Chevalley) is a linear transformation with one eigenvalue 1 and one eigenvalue $\neq 1$; see Sturmfels [9]. We are interested in finite groups generated by reflections, hence the second eigenvalue is necessarily a nontrivial root ζ of unity. Up to an invertible linear transformation, such a reflection is given by

$$T = \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Lemma 4. *Let T be a reflection of the plane, and let the nontrivial eigenvalue ζ be a primitive m^{th} root of unity. Then:*

(a) *T acts on polynomial functions via $f \mapsto f \circ T$. The space of polynomial functions is a direct sum of eigenspaces for this action, with eigenvalues $1, \zeta, \ldots, \zeta^{m-1}$.*
(b) *T acts on polynomial vector fields via $X \mapsto T^{-1}X \circ T$. The space of polynomial vector fields is a direct sum of eigenspaces for this action, with eigenvalues $1, \zeta, \ldots, \zeta^{m-1}$.*
(c) *If f is a polynomial such that $f \circ T = \zeta^r f$ then $T^{-1}X_f \circ T = \zeta^{r-1}X_f$.*

Proof. Parts (a) and (b) are immediate when T is given in the form (10), and this is sufficient for the proof. Part (c) follows from Lemma 2 (a) with $\Phi = T$. □

Let G be a finite reflection group in the plane. A characteristic property of reflection groups is that their invariant algebra admits an algebraically independent set of generators (Chevalley's theorem; see e.g. Sturmfels [9]). Thus in in the planar case there are two algebraically independent polynomials which generate the invariant algebra of G. Now consider a morphism Φ whose components are such generators for the invariant algebra. For instance, if G is generated by T as given in (10) then one may let $\Phi(x, y) = (x^m, y)$.

In the following we abbreviate $z := (x, y)^t$. Since Φ is built from invariants of G, we have

$$\Phi(Tz) = \Phi(z), \quad D\Phi(Tz)T = D\Phi(z), \quad \text{for all } T \in G. \tag{11}$$

Lemma 5. *Let G and Φ be as above. Then the following hold:*

(a) *Given a polynomial h, there exists a polynomial f such that $h = \widehat{f}$ if and only if $h \circ T = h$ for all $T \in G$.*

(b) *Given a vector field Y, there exists a vector field X such that $Y = \widehat{X}$ if and only if*
$$T^{-1}Y \circ T = (\det T)^{-1} \cdot Y,$$
for all $T \in G$.

(c) *If f is an irreducible polynomial such that $g := \widehat{f}$ is reducible, thus $g = g_1 \cdots g_s$ with irreducible g_i, then s divides the order m of G and the G-orbit of g_1 equals $\{g_1, \ldots, g_s\}$, up to multiplication by nonzero constants.*

Proof. (a) If $h = f \circ \Phi$ then for all z one has
$$h(Tz) = f(\Phi(Tz)) = f(\Phi(z)) = h(z),$$
whence $h \circ T = h$, for all $T \in G$. Conversely, $h \circ T = h$ for all T means that h is invariant for G, hence by construction of Φ there is an f such that $h = f \circ \Phi$. One direction of part (b) is obvious from (9) for the morphism T. For the converse direction assume that the identity holds and set
$$V(z) := (\det D\Phi(z))^{-1}\; D\Phi(z)Y(z).$$
A direct computation shows
$$T^{-1}V \circ T(z) = T^{-1}V(z),$$
for all $T \in G$ and all z. Therefore the entries of V are G-invariant and there is a vector field X such that $V = X \circ \Phi$. As for part (c), note that composition with T leaves g unchanged, and thus permutes the irreducible factors (up to nonzero constants). The product q of the elements in the orbit of (e.g.) g_1 satisfies $q \circ T = q$, and therefore $q = p \circ \Phi$ for some p. Moreover, p divides f because q divides g. Since f is irreducible, p must be a constant multiple of f, whence the orbit of g_1 contains all irreducible factors. □

Remark 6. One may use Lemma 5(c) to construct examples. Given a reflection group G, let p be an irreducible polynomial such that its G-orbit contains $|G|$ pairwise relatively prime polynomials $p = g_1, g_2, \ldots, g_m$. Then $g = \prod g_i$ is G-invariant by construction, hence $g = f \circ \phi$ for some polynomial f. Since the orbit of p has length $|G|$, f must be irreducible. If g satisfies the nondegeneracy conditions (ND1) and (ND2), Proposition 7 below is applicable for f.

The next result indicates that degeneracies for the underlying curves may restrict the possible nontrivial vector fields with a given integrating factor.

Proposition 7. *Let G and Φ be as above, and let f be an irreducible polynomial such that $\widehat{f} = g = g_1 \cdots g_s$ is reducible. Moreover, assume that g satisfies the nondegeneracy conditions (ND1) and (ND2). Then:*

(a) *The vector field X admits the integrating factor f^{-1} if and only if*
$$X = \alpha \cdot X_f + f \cdot \widetilde{X} \quad (\alpha \in \mathbb{C},\ \operatorname{div} \widetilde{X} = 0).$$

(b) *Given an integer $d > 1$, the vector field X admits the integrating factor f^{-d} if and only if*
$$X = f^d \left(\frac{\alpha}{f} \cdot X_f + Z_q^{(d)} \right)$$
for some $\alpha \in \mathbb{C}$ and some polynomial q.

Proof. By [3], Theorem 3, a vector field Y admits the integrating factor g^{-1} if and only if

$$Y = \sum_i \alpha_i \frac{g}{g_i} \cdot X_{g_i} + g \cdot X_h$$

with complex constants α_i and some polynomial h. The α_i and h are uniquely determined in this representation. (As for uniqueness, consider the case $Y = 0$ and check prime factors.) Now one verifies

$$T^{-1} Y \circ T = \sum_i \alpha_{\pi_T(i)} \frac{g}{g_i} \cdot X_{g_i} + g \cdot X_{h \circ T}$$

for all $T \in G$, where π_T is a permutation of the indices which is defined by $g_{\pi_T(i)} \circ T = g_i$.

Evaluating the condition $T^{-1} Y \circ T = (\det T)^{-1} \cdot Y$ from Lemma 5, using uniqueness and Lemma 2, shows that $h \circ T = h$ as well as $\alpha_{\pi_T(i)} = \alpha_i$ for all i, for every $T \in G$. Since the G-orbit of g_1 contains all prime factors of g, one finds $\alpha_1 = \cdots = \alpha_s$. The assertion of part (a) now follows from Lemma 2. Part (b) is a direct consequence of (a) and the reduction principle given in [3], Lemma 2 and Lemma 6. □

Remark 8. (a) The statement preceding the Proposition is to be understood as follows. Upon transfer to from f to $\hat{f} = g$ via Φ one obtains linear spaces $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}^0$. While the quotient space $\mathcal{F}/\mathcal{F}^0$ is one-dimensional, $\widehat{\mathcal{F}}/\widehat{\mathcal{F}}^0$ has dimension $s > 1$ (see [3]). In this sense, removing the degeneracy yields more nontrivial solutions to the inverse problem.

(b) Given a constant d which is not a positive integer, vector fields admitting the integrating factor g^{-d} may not always be of the form (3), although [3] indicates that exceptions are rare. But if a vector field Y admitting g^{-d} is of the form

$$Y = g^d \cdot Z_h^{(d)}$$

for some polynomial h, then there exists a vector field X such that $Y = \widehat{X}$ if and only if

$$X = f^d \cdot Z_p^{(d)}$$

for some polynomial p. To verify this, write $h = \sum h_\ell$ with $h_\ell \circ T = \zeta^\ell h_\ell$ and note that

$$T^{-1} Z_{h_\ell} \circ T = \zeta^{\ell - 1} Z_{h_\ell}$$

by Lemma 2(a). Therefore one may conclude $\mathcal{F} = \mathcal{F}^0$ if the corresponding property holds after transformation by Φ.

(c) One can employ the argument in the proof to construct all vector fields admitting f: Let

$$W = \sum_i a_i \frac{g}{g_i} \cdot X_{g_i}$$

with the property that $a_{\pi_T(i)} \circ T = a_i$ for all i. Then $T^{-1} W \circ T = (\det T)^{-1} \cdot W$ holds for all $T \in G$, and therefore $W = \widehat{Z}$ for some Z. If one chooses a_1 that is not G-invariant then one will obtain vector fields admitting f which are not contained in $\mathcal{V}^0 (= \mathcal{V}^1)$ (see [2] for notation and details). This simple direct construction of all nontrivial vector fields admitting f is also a particular feature of the morphisms related to reflection groups.

Example 1. Given a nonconstant polynomial q in one variable, with simple roots $v_1, \ldots, v_m$ that are all different from 0, consider

$$f = y^2 - x \cdot q(x)^2.$$

The polynomial f is irreducible, e.g. by Eisenstein's criterion. Now let

$$\Phi : \mathbb{C}^2 \to \mathbb{C}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y \end{pmatrix}.$$

Then

$$g(x, y) := \hat{f}(x, y) = \Big(y - x \cdot q(x^2)\Big) \cdot \Big(y + x \cdot q(x^2)\Big) =: g_1 \cdot g_2$$

is reducible and the nondegeneracy conditions (ND1), (ND2) apply. This example fits into the general scheme of Remark 6 with the group G generated by the reflection T about the y-axis, and $p = g_1$.

We first discuss vector fields admitting f. The singular points are precisely the $z_i = (v_i, 0)$, $1 \leq i \leq m$, and from their Hessian we see that all these points have multiplicity 1. Hence the quotient space $\mathcal{V}/\mathcal{V}^0$ has dimension m, according to [2], Theorem 8. The vector field

$$W := -xg_2 \cdot X_{g_1} + xg_1 \cdot X_{g_2}$$

admits g and satisfies $T^{-1}W \circ T = -W$, hence is of the form $W = \widehat{Z}$ for some Z. (The construction uses Remark 8(b) with $a_1 = -x$.) A straightforward computation shows

$$Z = 2x \cdot q(x)\partial/\partial x + y(2x \cdot q'(x) + q(x))\partial/\partial y.$$

The cofactor of Z is equal to $4x \cdot q'(x) + 2q(x)$ and hence does not vanish at any singular point. Now the argument from [2], Theorem 8 and its proof shows that

$$\mathcal{V}_f = \mathcal{V}_f^0 + \{b \cdot Z;\, b \in \mathbb{C}[x, y]\}.$$

Let us turn to integrating factors. According to Proposition 7, for any positive integer d the vector field X admits the integrating factor f^{-d} if and only if

$$X = f^d \cdot \left(\frac{\alpha}{f} \cdot X_f + Z_q^{(d)}\right)$$

with some constant α and some polynomial q.

Finally, given a constant d which is not a positive integer, the vector field X admits the integrating factor f^{-d} if and only if

$$X = f^d \cdot Z_p^{(d)}$$

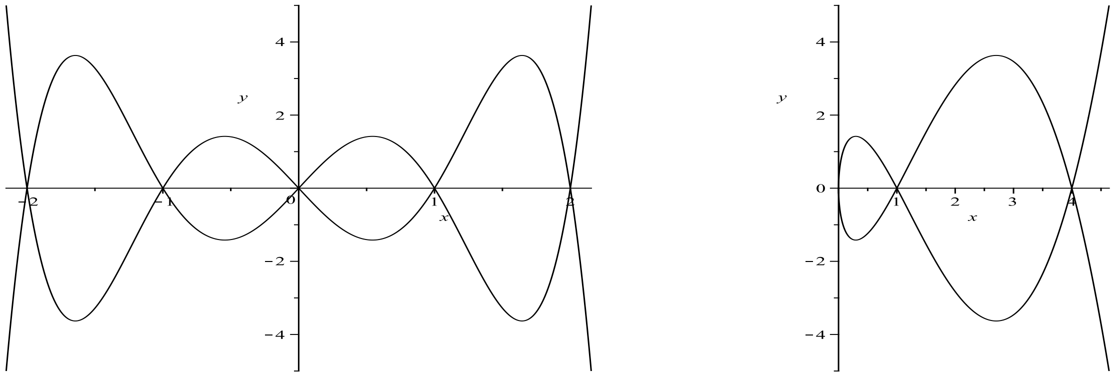
for some polynomial p. To see this we note that a vector field Y admits the integrating factor g^{-d} if and only if

$$Y = g^d \cdot Z_h^{(d)}$$

for some polynomial h, due to Theorem 24(b) of [3]. Remark 8 now shows that h is T-invariant.

For the purpose of illustration we consider a concrete example, with
$q(x) = (1 - x)(4 - x)$.

Figure 1 shows the zero set of the reducible polynomial g on the left, and the zero set of the irreducible polynomial f on the right, which is just the image of the former with respect to Φ.



$g = (y - x(1-x^2)(4-x^2))(y + x(1-x^2)(4-x^2))$. $f = y^2 - x(1-x)^2(4-x)^2$.

FIGURE 1. The zero sets of the reducible polynomial g and the irreducible polynomial f.

Example 2. Let q_1, q_2 be nonconstant polynomials in one variable with $q_1(0) \neq 0$, $q_2(0) \neq 0$, and

$$p = y + q_2(y^2) - xq_1(x^2).$$

Moreover let G be the four-element group generated by the reflections T_1 about the y-axis and T_2 about the x-axis. According to Lemma 5(c) and Remark 6, let $g_1 = p$, moreover

$$g_2 = y + q_2(y^2) + xq_1(x^2), \quad g_3 = -y + q_2(y^2) - xq_1(x^2), \quad g_4 = -y + q_2(y^2) + xq_1(x^2),$$

and define

$$g = g_1 \cdots g_4.$$

With

$$\Phi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

one obtains the irreducible polynomial

$$f = (y - q_2(y)^2)^2 - xq_1(x^2)(y + q_2(y)^2) + x^2 q_1(x)^4,$$

with $g = f \circ \Phi$. Now fix q_1 and assume that it has only simple roots, hence $x \mapsto x \cdot q_1(x^2)$ has only simple roots. Write $q_2 = v + \beta$ with a constant $\beta \neq 0$ and $v(0) = 0$. Then there are only finitely many values of β such that the curve $\{p = 0\}$ is not smooth, because the gradient of p has only finitely many zeros. Similarly, there are only finitely many values of β such that $y \mapsto q_2(y^2)$ or $y \mapsto y + q_2(y^2)$ or $y \mapsto y - q_2(y^2)$ has a multiple root. Excluding these exceptional values, one verifies by straightforward computation: The only singular points on the curve $\{g = 0\}$ are intersection points of two curves $\{g_i = 0\}$. There are no triple intersections, and the intersections are transversal. Thus, with the exception of finitely many values for β, Proposition 7 is applicable, and we have found, in a quite complicated-looking geometric setting, all the vector fields admitting the integrating factor f^{-d}, with d a positive integer. The case when d is not a positive integer depends on more specific properties of the reducible polynomial g.

As for vector fields admitting f, by Remark 8(b) we define

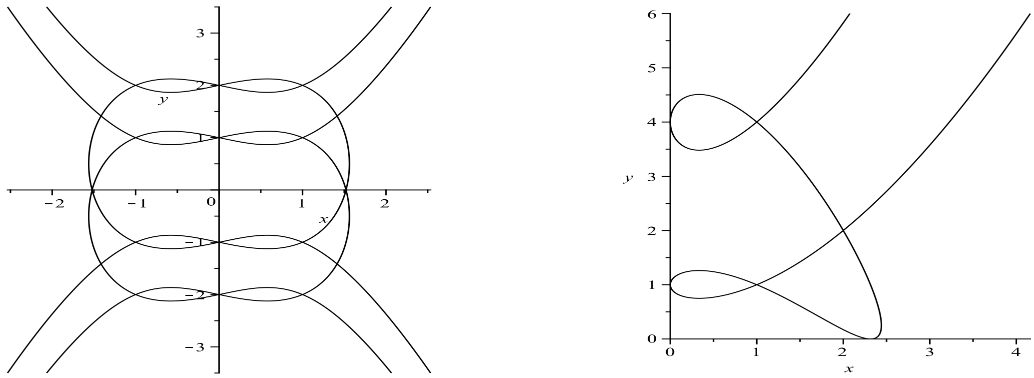
$$W_1 := \frac{xg}{g_1} \cdot X_{g_1} - \frac{xg}{g_2} \cdot X_{g_2} + \frac{xg}{g_3} \cdot X_{g_3} - \frac{xg}{g_4} \cdot X_{g_4},$$

and

$$W_2 := \frac{yg}{g_1} \cdot X_{g_1} + \frac{yg}{g_2} \cdot X_{g_2} - \frac{yg}{g_3} \cdot X_{g_3} - \frac{yg}{g_4} \cdot X_{g_4},$$

with $a_1 = x$ resp. $a_1 = y$. Then $W_i = \widehat{Z}_i$ for suitable Z_i, and an elementary verification similar to the one in Example 1 shows that $\mathcal{V}$ is spanned by $\mathcal{V}^0$ and polynomial multiples of the Z_i.

Again we consider a concrete example for illustration, with $q_1(x) = 1-x$ and $q_2(y) = y - 2$.



The graph of $g_1g_2g_3g_4 = g = 0$. The graph of $f = 0$.

FIGURE 2. The zero set of the reducible polynomial g and the irreducible polynomial f. Note that $g(x, y) = \hat{f}(x, y)$.

Figure 2 shows the zero set of the reducible polynomial g on the left, and the zero set of the irreducible polynomial f on the right, which is just the image of the former with respect to Φ.

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