Orbit space reduction and localizations

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Abstract

We review the familiar method of reducing a symmetric ordinary differential equation via invariants of the symmetry group. Working exclusively with polynomial invariants is problematic: Generator systems of the polynomial invariant algebra, as well as generator systems for the ideal of their relations, may be prohibitively large, which makes reduction unfeasible. In the present paper we propose an alternative approach which starts from a characterization of common invariant sets of all vector fields with a given symmetry group, and uses suitably chosen localizations. We prove that there exists a reduction to an algebraic variety of codimension at most two in its ambient space. Some examples illustrate the approach.

Key words: Symmetry; invariant theory; algebraic group.
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1 Introduction

In this note we consider a symmetric ordinary differential equation

$$\dot{x} = f(x) \tag{1}$$

on $\mathbb{R}^n$ or $\mathbb{C}^n$, with the symmetries forming a subgroup G of the general linear group $GL(n)$. There is an obvious benefit to be gained from symmetries, as one may employ them to find new solutions from given ones. Moreover there is a less obvious (but also familiar) benefit, since one may employ symmetry reduction. As representatives of the many publications on the subject we mention only Field [7] (for compact groups), and Cushman and Bates [4] (for reductive groups).

In the present paper we first focus on the conceptually straightforward approach to reduction via the Hilbert map which is constructed from polynomial invariants of G. (To avoid further technicalities we will discuss only polynomial vector fields here.) The survey article by Chossat [3] provides a very good introduction to this method for compact G, as well as a number of examples. Chossat also points out the limitations of the approach: There may be problems with its feasibility, since (even minimal) generator systems of polynomial invariants may be very large. The main purpose of the present paper is to suggest a possible escape from such feasibility problems. Roughly speaking, we will introduce a refinement of orbit space reduction via polynomials by introducing carefully chosen denominators to achieve a reduction via rational functions. Using a theorem due to Grosshans [8], we show that the necessary number of generators is at most two higher than the number dictated by dimensions of group orbits. In our proofs we will make use of some results and tools from commutative algebra and elementary algebraic geometry. A few examples illlustrate the reduction method.

2 An overview of the orbit space method

2.1 Blanket assumptions and notation

We first introduce some notions and hypotheses which will be kept throughout the paper. Some of the assumptions as stated are more restrictive than necessary; our focus is on reduction mechanisms and we want to keep technicalities to a minumum.

- $\mathbb{K}$ stands as an abbreviation for $\mathbb{R}$ or $\mathbb{C}$.
- $G \subseteq GL(n, \mathbb{K})$ is a linear algebraic group (i.e. a subgroup of $GL(n, \mathbb{K})$ defined by polynomial equations) which acts naturally on $\mathbb{K}^n$.
- Furthermore the orbits of G have generic dimension $s > 0$.
- We restrict attention to the case of polynomial vector fields, thus f has polynomial entries (unless specified otherwise).
- The differential equation (1) is symmetric (equivariant) with respect to G; i.e., $T^{-1}fT = f$ for all $T \in G$.

Recall that a polynomial $\psi \in \mathbb{K}[x_1, \ldots, x_n]$ is called G-invariant if $\psi \circ T = \psi$ for all $T \in G$. The G-invariant polynomials form a subalgebra of the polynomial algebra which is denoted $\mathbb{K}[x_1, \ldots, x_n]^G$. We add one more hypothesis.

- We assume that $\mathbb{K}[x_1, \ldots, x_n]^G \neq \mathbb{K}$ admits a finite set $\gamma_1, \ldots, \gamma_r$ of generators.
- Accordingly we define the Hilbert map

$$\Gamma := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{pmatrix} : \mathbb{K}^n \to \mathbb{K}^r.$$

The choice of G as a linear group is not as restrictive as it may initially seem: There are local linearization theorems for compact groups (Bochner; see e.g. Duistermaat and Kolk [6]) and for semisimple Lie groups (see e.g. Kushnirenko [13]). The choice of f as a polynomial is restrictive, but one may think of Taylor expansions and results e.g. by Schwarz [21], Luna [15] and Poenaru [19] which guarantee extensions to the smooth and analytic case.

2.2 The basic reduction mechanism

There is one more notion we need to introduce.

Definition 1. *The* Lie derivative *of a scalar valued function ϕ with respect to a vector field g is defined by*

$$\phi \mapsto L_g(\phi); \quad L_g(\phi)(x) := D\phi(x)g(x).$$

The Lie derivative describes the rate of change of ϕ along solutions of the differential equation $\dot{x} = g(x)$. A fundamental fact (with straightforward proof) is the following.

Lemma 1. *If the differential equation* (1) *is G-symmetric and ψ is G-invariant then $L_f(\psi)$ is G-invariant.*

From this observation one obtains a reduction map as follows.

Proposition 1. *If $\mathbb{K}[x_1, \ldots, x_n]^G$ admits the finite set $\gamma_1, \ldots, \gamma_r$ of generators then the* Hilbert map $\Gamma = (\gamma_1, \ldots, \gamma_r)^{\mathrm{tr}}$ *sends the G-symmetric vector field f to some vector field h on $\mathbb{K}^r$. The equation $\dot{x} = h(x)$ admits as an invariant set the algebraic variety $Y := \overline{\Gamma(\mathbb{K}^r)}$ (Zariski closure) which is defined by the polynomial relations between $\gamma_1, \ldots, \gamma_r$.*

Sketch of proof. By Lemma 1 there exist polynomials η_i in r variables such that

$$L_f(\gamma_i) = \eta_i(\gamma_1, \ldots, \gamma_r).$$

See e.g. [20] for more details. $\square$

This procedure is also known as *orbit space reduction*; see Chossat [3]. To explain the name, note that all invariants are constant on G-orbits. For compact groups there is a 1-1 correspondence between G-orbits and points in the image of the Hilbert map (see e.g. Field [7]). For more general classes of groups such a strict correspondence does not hold but Y is the "best possible quotient" of the group action that is still an algebraic variety; see Popov and Vinberg [18], subsection 4.3.

There is a fundamental a priori obstruction to this approach, since the polynomial invariant algebra of certain linear algebraic groups is not finitely generated; see Nagata [16]. (According to our blanket hypotheses we will not consider such groups.) This does not seem to be much of an obstacle in practice, since for many interesting cases (e.g. all semisimple and, more generally, all reductive groups) finite generation of $\mathbb{K}[x_1, \ldots, x_n]^G$ is guaranteed. On the other hand there is a practical obstruction which may be quite annoying: The minimal number of generators for $\mathbb{K}[x_1, \ldots, x_n]^G$ may be quite large even if the group is reductive and the action seems harmless.

Example 1. *(a) Let $m \in \mathbb{N}$, E_m the $m \times m$ identity matrix and*

$$G = \left\{ \begin{pmatrix} a \cdot E_m & 0 \\ 0 & a^{-1} \cdot E_m \end{pmatrix}; a \in \mathbb{K}^* \right\} \subseteq GL(2m, \mathbb{K}).$$

One can verify that the invariant algebra admits the smallest generator set

$$\{\gamma_{ij} := x_i x_{m+j};\ 1 \leq i,\ j \leq m\}$$

with m^2 elements. There are many relations between these generators, viz.

$$\gamma_{ij} \cdot \gamma_{k\ell} = \gamma_{i\ell} \cdot \gamma_{kj}.$$

One can show that these, with all indices running from 1 to m, actually generate the ideal of relations. Thus one cannot avoid dealing with a high dimensional embedding space, and with a rather complicated image of the Hilbert map.
(Note that for $\mathbb{K} = \mathbb{C}$ this group is the complexification of the diagonal action of $SO(2, \mathbb{R})$ on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$.)

(b) *The irreducible* 11*-dimensional representation of* $SL(2,\mathbb{C})$ *requires a generator system with 106 elements; see Brouwer and Popoviciu [2]; thus the embedding space of the variety* Y *(which has dimension* 8*) is 106-dimensional. No relations between the generators are given in [2].*

Some general remarks on constructive invariant theory and estimates can be found in Derksen and Kraft [5].

2.3 Rational reduction

A possible alternative is to consider the field $\mathbb{K}(x_1,\ldots,x_n)^G$ of rational invariants. The following fact is well-known; see e.g. Popov and Vinberg [18] §2.

Proposition 2. *The field* $\mathbb{K}(x_1,\ldots,x_n)^G$ *has transcendence degree* $n-s$ *over* $\mathbb{K}$*, with* s *the generic orbit dimension of the group action. It admits a generator set with at most* $n-s+1$ *elements.*

Sketch of proof. We refer e.g. to Lang [14] for the necessary results from algebra. A subfield of a finitely generated extension of $\mathbb{K}$ is also finitely generated. The transcendence degree of $\mathbb{K}(x_1,\ldots,x_n)^G$ is equal to $n-s$; see Borel [1], Ch. AG.10. Let $\psi_1,\ldots,\psi_{n-s}$ be a transcendence basis of the invariant field. Then $\mathbb{K}(x_1,\ldots,x_n)^G$ is a finitely generated (hence finite) algebraic extension of $\mathbb{K}(\psi_1,\ldots,\psi_{n-s})$, which can be generated by a single element due to the primitive element theorem. □

Remark 1. *The computation of generators for* $\mathbb{K}(x_1,\ldots,x_n)^G$ *is algorithmically accessible; see Hubert and Kogan [11].*

Example 2. *We continue Example 1(a), thus*

$$G=\left\{\begin{pmatrix} a\cdot E_m & 0\\ 0 & a^{-1}\cdot E_m\end{pmatrix}\right\}$$

with generic orbit dimension one. Here the $2m-1$ *generators*

$$\begin{array}{llll}\gamma_{1,1}, & \gamma_{1,2}, & \ldots, & \gamma_{1,m},\\ \gamma_{2,1}, & \gamma_{3,2}, & \ldots, & \gamma_{m,m-1}\end{array}$$

suffice, due to $\gamma_{i,j}=\gamma_{i,i-1}\gamma_{1,j}/\gamma_{1,i-1}$.

Remark 2. *We want to exhibit one class of groups for which the invariant algebra and the invariant field are relatively easy to characterize; for the sake* *of simplicity we restrict attention to complex groups. An* algebraic torus *is a connected diagonal (more generally, a diagonalizable) subgroup of some* $GL(n,\mathbb{C})$*; see e.g. Humphreys [12], Section 16. (Example 2 falls in this class.) For an algebraic torus the polynomial invariant algebra is generated by invariant monomials, and the field of invariants is generated by invariant Laurent monomials. Pars pro toto, we verify this fact for a one-parameter torus. Thus let*

$$d_1,\ldots,d_n\in\mathbb{Z};\quad T_a:=\operatorname{diag}\left(a^{d_1},\ldots,a^{d_n}\right)\ \textit{for}\ a\neq 0;\quad G:=\{T_a;\,a\in\mathbb{C}^*\}\,.$$

For a Laurent monomial $\phi:=x_1^{m_1}\cdots x_n^{m_n}$ *one computes*

$$\phi(T_a x)=\phi(x)\cdot a^{\sum d_i m_i},$$

hence each Laurent monomial lies in an eigenspace of the corresponding action of T_a*, with eigenvalue* $a^{\sum d_i m_i}$*. Therefore polynomial invariants must be linear combinations of invariant monomials; for rational invariants one shows with a bit more effort that they are quotients of linear combinations of Laurent monomials. In either case the condition for invariance is*

$$\sum d_i m_i=0.$$

Over the integers this condition (with fixed d_i*) defines a* $\mathbb{Z}$*-module which is free of rank* $n-1$*, thus has a basis with* $n-1$ *elements (which are easy to compute). The corresponding Laurent monomials generate the invariant field* $\mathbb{C}(x_1,\ldots,x_n)^G$*, which is purely transcendental. (With regard to invariant polynomials, the conditions remain the same but now only nonnegative integer solutions are admissible, which makes theory and computations harder. However, the invariant algebra is generated by finitely many monomials; see e.g. [22].)*

From the rational analogue of the Hilbert map one obtains a reduction method for symmetric vector fields; the closure of the image is $\mathbb{K}^{n-s}$ or a subvariety of codimension one in $\mathbb{K}^{n-s+1}$. The problem lies in the behavior of denominators, whence the domain of definition of a Hilbert map is just an open subset of $\mathbb{K}^n$. Even worse, one cannot guarantee the existence of a reduction near some specific point, and one generally loses information about interesting invariant sets (such as lower-dimensional strata in the compact group case) that are common to all G-symmetric vector fields. Therefore some kind of middle ground between rational reduction via generators of the invariant field $\mathbb{K}\left(x_1,\ldots,x_n\right)^G$ on the one hand, and polynomial reduction via generators of the invariant algebra $\mathbb{K}\left[x_1,\ldots,x_n\right]^G$ on the other hand would be preferable.

3 Invariant sets

As noted above, common invariant sets of all G-symmetric vector fields (which are called *invariant sets forced by symmetry* in [10]) are of particular interest for applications. From a different perspective, such invariant sets may also be seen as a cause for the problems with reduction by polynomial invariants, viz., the high dimension of the embedding space for the variety $Y = \overline{\Gamma(\mathbb{K}^n)}$.

Example 3. *We continue Example 2, with*

$$G = \left\{ \begin{pmatrix} a \cdot E_m & 0 \\ 0 & a^{-1} \cdot E_m \end{pmatrix}; a \in \mathbb{K}^* \right\} \subseteq GL(2m, \mathbb{K}).$$

The closure Y of the image of the Hilbert map in the m^2 dimensional affine space with (conveniently chosen) coordinates y_{ij}, $1 \le i, j \le m$, is defined by the equations

$$y_{ij} \cdot y_{k\ell} - y_{i\ell} \cdot y_{kj} = 0.$$

Since $0 \in Y$ and all the defining polynomials have zero derivative at 0, the tangent space to Y at 0 (in the sense of algebraic geometry, see Humphreys [12], Ch. I, §5) has dimension m^2. Therefore it is impossible to embed Y into an affine space of smaller dimension. (Since any other image of $\mathbb{K}^n$ under a Hilbert map induced by G is isomorphic to Y, a different choice of generators cannot remedy this problem.)

Thus a closer look at the singular points of Y seems in order. The set of singular points of Y is invariant for the reduced vector field (see e.g. [22], Prop. 3.11), and its inverse image in $\mathbb{K}^n$ is an invariant set of the original symmetric differential equation (1). Therefore we now focus on common invariant sets of G-symmetric vector fields.

Definition 2. *(a) We denote by $\mathcal{D}_G$ the set of all G-symmetric polynomial vector fields on $\mathbb{K}^n$.*

(b) A set $Y \subseteq \mathbb{K}^n$ is called $\mathcal{D}_G$-invariant if it is an invariant set for every G-symmetric polynomial differential equation.

One easily verifies that $\mathcal{D}_G$ is a Lie algebra with the usual compositions. Moreover one sees that unions, intersections and complements of $\mathcal{D}_G$-invariant sets are $\mathcal{D}_G$-invariant. In particular, for every $v \in \mathbb{K}^n$ there exists a minimal $\mathcal{D}_G$-invariant set containing v. By continuous dependence properties the closure, boundary and interior (with respect to the norm

topology) of a $\mathcal{D}_G$-invariant set is $\mathcal{D}_G$-invariant. Moreover, every connected component (with respect to the norm topology) of a $\mathcal{D}_G$-invariant set is obviously $\mathcal{D}_G$-invariant.
The following results (some of which are commonly known) are quoted from [10], where proofs are given when necessary. We start with some basic observations that also establish a connection to the familiar stratification of orbit space for compact groups.

Proposition 3. *(a) Let H be a (closed) subgroup of G. Then every $\mathcal{D}_H$-invariant set is also $\mathcal{D}_G$-invariant.*

(b) Given $f \in \mathcal{D}_G$, all points on a trajectory of $\dot{x} = f(x)$ have the same isotropy subgroup.

(c) Given any (closed) subgroup H of G, the fixed point subspace

$$\mathrm{Fix}(H) := \{z : Tz = z \textit{ for all } T \in H\}$$

is $\mathcal{D}_G$-invariant. In particular, for $v \in \mathbb{K}^n$ the fixed point space $\mathrm{Fix}(G_v)$ *of its isotropy group is $\mathcal{D}_G$-invariant.*

We note that for compact groups, and more generally for proper group actions (see e.g. Cushman and Bates [4]), singular points of $\Gamma(\mathbb{R}^n)$ correspond to points in $\mathbb{R}^n$ with an isotropy group that is not minimal.
For a comprehensive characterization of $\mathcal{D}_G$-invariant sets the following notions are useful.

Definition 3. *(a) The* evaluation map *at $v \in \mathbb{K}^n$ is defined by*

$$\epsilon_v : \mathcal{D}_G \to \mathbb{K}^n, \quad f \mapsto f(v).$$

(b) For a nonnegative integer m let

$$Z_m = Z_m(G) := \{y \in \mathbb{K}^n : \dim(\epsilon_y(\mathcal{D}_G)) \leq m\} \textit{ and } Z^*_{m+1} := Z_{m+1} \backslash Z_m.$$

Now we can state the principal results.

Theorem 1. *(a) For every $y \in \mathbb{K}^n$ the subspace $\epsilon_y(\mathcal{D}_G)$ is $\mathcal{D}_G$-invariant.*

(b) For every $m \geq 0$ the sets Z_m and Z^*_{m+1} are $\mathcal{D}_G$-invariant, and also stable with respect to the group action.

Theorem 2. *Let $y \in Z^*_m$. The smallest $\mathcal{D}_G$-invariant set containing y is the connected component, in the norm topology, of y in $\epsilon_y(\mathcal{D}_G) \setminus Z_{m-1}$. Moreover, $\epsilon_y(\mathcal{D}_G)$ is the smallest Zariski-closed $\mathcal{D}_G$-invariant set which contains y.*

Theorem 3. *One has $Z_n^* = \emptyset$ if and only if there exists a nonconstant rational function that is a first integral for all elements of $\mathcal{D}_G$ on $\mathbb{K}^n$.*

Remark 3. *For reductive G, these results carry over to vector fields on arbitrary affine subvarieties that are G-stable (i.e., are mapped to themselves by any transformation in G). The underlying reason is that every invariant of the G-action on the subvariety is the restriction of some element of $\mathbb{K}[x_1, \ldots, x_n]^G$ (see e.g. Panyushev [17]). By the same token every G-symmetric vector field on the subvariety is the restriction of some element of $\mathcal{D}_G$.*

For special classes of groups one can determine the minimal Zariski-closed $\mathcal{D}_G$-invariant subsets. We collect some results in a rather informal manner.

Examples. *(a)* Groups of diagonal matrices: *All minimal Zariski-closed $\mathcal{D}_G$-invariant sets are coordinate subspaces, thus equal to the common zero set of certain coordinate functions $x_{i_1}, \ldots, x_{i_m}$. There is a precise description (depending on the characters of the group) of those coordinate subspaces which are $\mathcal{D}_G$-invariant.*

(b) Compact groups: *The familiar stratification (see e.g. Field [7]) is recovered; all minimal Z-closed $\mathcal{D}_G$-invariant sets are isotropy fixed point spaces.*

(c) Reductive groups: *In case $\mathbb{K} = \mathbb{C}$ the set $\epsilon_y(\mathcal{D}_G)$ is equal to the isotropy fixed point space whenever the G-orbit of y is closed. Matters are less clear for non-closed orbits; here one should note recent work by Grosshans and Kraft [9]. In case $\mathbb{K} = \mathbb{R}$ the statement regarding closed orbits remains valid, but one has to consider the isotropy fixed point space in the complexification and intersect this with $\mathbb{R}^n$.*

4 Reduction via localizations

4.1 Motivation

The dilemma with orbit space reduction, as evident from Section 2, is twofold. Reduction via polynomial invariants is often not manageable, but reduction via rational invariants may be too coarse. Using localizations seems to open a feasible alternative.

Denote by Q be the quotient field of $\mathbb{K}[x_1, \ldots, x_n]^G$ and let q be its transcendence degree over $\mathbb{K}$. Then the dimension of $Y = \overline{\Gamma(\mathbb{K}^n)}$ is equal to q

(see e.g. Borel [1], Ch. AG.10), hence one needs at least q generators for $\mathbb{K}[x_1,\ldots,x_n]^G$. By the primitive element theorem one sees that at most $q+1$ suitable elements generate Q, and there exists a rational Hilbert map to $\mathbb{K}^{q+1}$. Grosshans [8] proved that there exists a suitable denominator $\psi \in \mathbb{K}[x_1,\ldots,x_n]^G$ such that at most $q+3$ generators (including ψ) suffice for $\mathbb{K}[x_1,\ldots,x_n]^G\left[\frac{1}{\psi}\right]$. The corresponding Hilbert map is polynomial and sends $\mathbb{K}^n$ to a q-dimensional subvariety of $\mathbb{K}^{q+3}$. Thus one could say that one needs at most three generators in excess of what the transcendence degree of Q dictates. For our purpose, Grosshans' result needs some refinement, since we want to keep control over the points where such a reduction is possible (in other words, where there is some denominator which does not vanish at the point in question). Moreover we establish a slight (but welcome) improvement concerning the number of generators by showing that $q+2$ suitable generators suffice.

4.2 The main result

From now on we assume that $Z_n^* \neq \emptyset$, so there exist points with surjective evaluation map. We will show that there exists a "suitable denominator" for reduction at each point in Z_n^*.

Proposition 4. *Denote by Q be the quotient field of $\mathbb{K}[x_1,\ldots,x_n]^G$ and let q be its transcendence degree over $\mathbb{K}$.*

(a) For any $m \in \mathbb{N}$ the set

$$Y_m := \{y \in \mathbb{K}^n : \mathit{rank}\, D\Gamma(y) \leq m\}$$

is $\mathcal{D}_G$-invariant.

(b) The maximal rank of $D\Gamma(y)$, $y \in \mathbb{K}^n$, equals q, and $\mathit{rank}\, D\Gamma(v) = q$ for every $v \in Z_n^*$.

Proof. (a) We include a proof of this known result (see e.g. [23], Prop. 3.4) for the reader's convenience. Let $f \in \mathcal{D}_G$. Γ is solution-preserving from $\dot{x} = f(x)$ to $\dot{x} = h(x)$ for some vector field h on $\mathbb{K}^r$. Therefore the equation

$$\Gamma(F(t,y)) = H(t,\Gamma(y))$$

holds, with $F(t,y)$ and $H(t,\Gamma(y))$ being the solutions of the initial value problems $\dot{x}=f(x)$, $x(0)=y$ and $\dot{x}=h(x)$, $x(0)=\Gamma(y)$, respectively. Differentiating with respect to y, one finds

$$D\Gamma(F(t,y))D_2F(t,y)=D_2H(t,\Gamma(y))D\Gamma(y).$$

Since $D_2F(t,y)$ and $D_2H(t,\Gamma(y))$ are invertible (being solutions of the variational equations associated to $\dot{x}=f(x)$ resp. $\dot{x}=h(x)$), $D\Gamma(F(t,y))$ and $D\Gamma(y)$ have the same rank for all t. This argument actually shows that $\{y\in\mathbb{K}^n:\,\mathrm{rank}\,D\Gamma(y)=m\}$ is $\mathcal{D}_G$-invariant.

(b) By general properties of morphisms of algebraic varieties (see e.g. Borel [1] Ch. AG.17), one has that $\mathrm{rank}\,D\Gamma(y)\le q$ for all $y\in\mathbb{K}^n$, with equality holding on a nonempty Zariski open set.
Let $v\in Z_n^*$. Suppose that

$$v\in Y_{q-1}=\{y\in\mathbb{K}^n:\,\mathrm{rank}\,D\Gamma(y)\le q-1\}.$$

Y_{q-1} is $\mathcal{D}_G$-invariant by part (a), and Zariski-closed. The smallest Zariski-closed $\mathcal{D}_G$-invariant set which contains v is $\epsilon_v(\mathcal{D}_G)$, thus $\epsilon_v(\mathcal{D}_G)\subseteq Y_{q-1}$. Because $v\in Z_n^*$, we have $\mathbb{K}^n=\epsilon_v(\mathcal{D}_G)=Y_{q-1}$; a contradiction to $Y_q\setminus Y_{q-1}\neq\emptyset$. Thus $v\in Y_q\setminus Y_{q-1}$.

□

Remark 4. *Since Q is a subfield of $\mathbb{K}(x_1,\ldots,x_n)^G$, the transcendence degree q is at most equal to $n-s$, with s the generic orbit dimension of the group action. For many group actions one has $Q=\mathbb{K}(x_1,\ldots,x_n)^G$. Examples include algebraic one-parameter tori which admit nonconstant polynomial invariants, and also groups which admit no nontrivial homomorphism to $\mathbb{C}^*$ (such as simple groups).*

Next we show that suitable localizations yield a feasible local reduction method near any point of Z_n^*. The proof builds on the work of Grosshans [8] (specifically the Theorem in Section 3 and Theorem 1 in Section 4). For our particular setting we need to adjust some arguments.

Theorem 4. *Denote by Q the quotient field of $\mathbb{K}[x_1,\ldots,x_n]^G$ and let q be its transcendence degree over $\mathbb{K}$. For any $v\in Z_n^*$ there exist an integer ℓ with $q\le\ell\le q+1$ and $\psi,\psi_1,\ldots,\psi_\ell\in\mathbb{K}[x_1,\ldots,x_n]^G$ such that*

$$\mathbb{K}[x_1,\ldots,x_n]^G\left[\frac{1}{\psi}\right]=\mathbb{K}[\psi_1,\ldots,\psi_\ell]\left[\frac{1}{\psi}\right]\quad\textit{and}\quad\psi(v)\neq 0.$$

(The possibility that $\psi\in\mathbb{K}^*$ is included in this statement.)

Proof. We first discuss the case $\mathbb{K} = \mathbb{C}$. By Proposition 4, $D\Gamma(v)$ has rank q, hence there exist q of the γ_i (denoted by $\psi_1, \ldots, \psi_q$) such that $\widetilde{\Psi} := (\psi_1, \ldots, \psi_q)^{\mathrm{tr}}$ satisfies $\operatorname{rank} D\widetilde{\Psi}(v) = q$, and $\psi_1, \ldots, \psi_q$ form a transcendence basis of Q. If $Q = \mathbb{K}(\psi_1, \ldots, \psi_q)$ then we are done. Otherwise, since Q is an algebraic extension of $\mathbb{K}(\psi_1, \ldots, \psi_q)$ which is generated by $\gamma_1, \ldots, \gamma_r$, there exists a polynomial generator for Q of the form $\sum c_i \gamma_i$ with integer coefficients c_i; see the argument in the proof of the primitive element theorem V.4.6 in Lang [14]. In conclusion, Q is generated by polynomial invariants $\psi_1, \ldots, \psi_\ell$, and for $\Psi := (\psi_1, \ldots, \psi_\ell)^{\mathrm{tr}}$ one has $\operatorname{rank} D\Psi(v) = q$.

Let $\widehat{Y}$ be the Zariski closure of $\Psi(\mathbb{C}^n)$; this is an irreducible affine variety. For $v \in Z_n^*$ we first show that there is a neighborhood (with respect to the norm topology) $\widetilde{U}$ of v in $\mathbb{C}^n$ such that $\Psi(\widetilde{U})$ is a neighborhood (with respect to the relative topology induced by the norm topology) of $\Psi(v)$ in $\widehat{Y}$. To verify this, recall that $\operatorname{rank} D\Psi(v) = q$, hence one may assume (by numbering the x_j appropriately) that $\left(\frac{\partial \psi_i}{\partial x_j}(v)\right)_{1 \le i \le \ell, 1 \le j \le q}$ has rank q. This implies that

$$\mathbb{K}^q \to \mathbb{K}^\ell, \quad (x_1, \ldots, x_q)^{\mathrm{tr}} \mapsto \Psi(x_1, \ldots, x_q, v_{q+1}, \ldots, v_n)$$

is an immersion near $(v_1, \ldots, v_q)^{\mathrm{tr}}$, with the image being a local analytic manifold of dimension q that contains $\Psi(v)$ and is contained in the irreducible q-dimensional variety $\widehat{Y}$. Therefore some ("relative norm topology") neighborhood of $\Psi(v)$ in $\widehat{Y}$ coincides with this local analytic manifold. We conclude that $\Psi(v)$ is not contained in the Zariski closure Y^* of $\widehat{Y} \setminus \Psi(\mathbb{C}^n)$, since the complement V of Y^* in $\widehat{Y}$ is open and dense with respect to the relative norm topology as well as the Zariski topology. (See e.g. Humphreys [12], Ch. I, 4.4.) Thus $\Psi(v) \in V$, and v is contained in the Zariski-open subset $U := \Psi^{-1}(V)$ of $\mathbb{C}^n$. The existence of the local immersion also shows that the tangent space to $\widehat{Y}$ at $\Psi(v)$ has dimension q, hence $\Psi(v)$ is a simple point.

Now the facts and arguments in [8] (Theorem in Section 3 and Theorem 1 in Section 4) can be applied verbatim to show the following.

(i) If $\Psi(U) = \widehat{Y}$, then $\Psi(\mathbb{C}^n) = \widehat{Y}$ and

$$\mathbb{C}[x_1, \ldots, x_n]^G = \mathbb{C}\left[\widehat{Y}\right] = \mathbb{C}[\psi_1, \ldots, \psi_\ell].$$

(ii) Otherwise, there exists a nonzero element $\rho \in \mathbb{C}\left[\widehat{Y}\right]$ such that ρ vanishes on $\widehat{Y} \setminus \Psi(U)$ and $\rho(\Psi(v)) \neq 0$. Thus there is a nonzero element

$\psi := \rho \circ \Psi$ in $\mathbb{C}[x_1, \dots, x_n]^G$ such that ψ vanishes on $\mathbb{C}^n \setminus U$ and $\psi(v) \neq 0$. Then

$$\mathbb{C}[x_1, \dots, x_n]^G \left[\frac{1}{\psi}\right] = \mathbb{C}[\psi_1, \dots, \psi_\ell] \left[\frac{1}{\psi}\right].$$

This finishes the proof for the complex case.

For $\mathbb{K} = \mathbb{R}$, the generators γ_i and ψ_j may be taken in $\mathbb{R}[x_1, \dots, x_n]$. Complexification yields $\widehat{Y}$ being defined over $\mathbb{R}$. Since $v \in \mathbb{R}^n$, whenever ρ does not vanish at $\Psi(v)$ then the same holds for the complex conjugate $\overline{\rho}$, and one may proceed with $\sigma := \rho \cdot \overline{\rho}$ instead of ρ for the rest of the argument. $\square$

Remark 5. *Relations satisfied by the ψ_i and ψ can (in principle) be obtained from the proofs. If $Q \neq \mathbb{K}(\psi_1, \dots, \psi_q)$ then the minimum polynomial of ψ_{q+1} provides a nontrivial relation between $\psi_1 \dots, \psi_{q+1}$. Moreover, if $\Psi(U) \neq \widehat{Y}$ then by the last step (item (ii)) in the proof of Theorem 4 one has*

$$\psi = \rho(\psi_1, \dots, \psi_\ell) \in \mathbb{K}[\psi_1, \dots, \psi_\ell].$$

Corollary 1. *Let the setting and notation of Theorem 4 be given.*

(a) The map

$$\widehat{\Psi} : \mathbb{K}^n \to \mathbb{K}^{\ell+1}, \quad x \mapsto \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_\ell(x) \\ \psi(x) \end{pmatrix}$$

sends any G-symmetric polynomial vector field f to a vector field h with entries in $\mathbb{K}[y_1, \dots, y_\ell, y_{\ell+1}][1/y_{\ell+1}]$, and h is regular at $\widehat{\Psi}(v)$.

(b) If $\psi \in \mathbb{K}[\psi_1, \dots, \psi_\ell]$, one may instead employ the reduction map

$$\widehat{\widehat{\Psi}} : \mathbb{K}^n \to \mathbb{K}^\ell, \quad x \mapsto \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_\ell(x) \end{pmatrix}.$$

The reduced vector field then has entries in $\mathbb{K}[y_1, \dots, y_\ell][1/\theta_0(y_1, \dots, y_\ell)]$, using the notation from Remark 5.

The image of $\widehat{\Psi}$ (resp.$\widehat{\widehat{\Psi}}$) may (in principle) be determined via Remark 5. Note that Z_n^* can be covered by finitely many Zariski open sets U_i such that reduction in the sense of the Corollary works on each U_i.

Example 4. *We specialize Example 3 to $m = 4$, i.e.*

$$G = \left\{ \begin{pmatrix} a \cdot E_4 & 0 \\ 0 & a^{-1} \cdot E_4 \end{pmatrix}; a \in \mathbb{K}^* \right\}.$$

One can show (using [10], Prop. 4.3) that

$$Z_7 = \left\{ x \in \mathbb{K}^8 \mid x_1 = x_2 = x_3 = x_4 = 0 \right\} \cup \left\{ x \in \mathbb{K}^8 \mid x_5 = x_6 = x_7 = x_8 = 0 \right\}.$$

We consider $v := (v_1, 0, 0, 0, v_5, 0, 0, 0) \in Z_8^*$.
Recall that the invariant algebra can be generated by the 16 elements

$$\gamma_{ij} := x_i x_{4+j};\ 1 \le i,\ j \le 4,$$

and let

$$\Psi := (x_1x_5, x_2x_5, x_3x_5, x_4x_5, x_1x_6, x_1x_7, x_1x_8)^{\mathrm{tr}}.$$

Invoking Remark 2 one verifies that the entries of Ψ already suffice to generate Q, which has transcendence degree 7 over $\mathbb{K}$. Since

$$D\Psi(v) = \begin{pmatrix} v_5 & 0 & v_1 & 0 \\ 0 & v_5 E_3 & 0 & 0 \\ 0 & 0 & 0 & v_1 E_3 \end{pmatrix}$$

has rank 7, we know from the proof that Ψ will work for the given v. One finds

$$\Psi(\mathbb{K}^8) = \left\{ y \in \mathbb{K}^7 \mid y_1 \neq 0 \ \vee\ y_1 = y_2 = y_3 = y_4 = 0 \ \vee\ y_1 = y_5 = y_6 = y_7 = 0 \right\}$$

with Zariski closure $\widehat{Y} = \mathbb{K}^7$. The Zariski closure of the complement of $\Psi(\mathbb{K}^8)$ in $\mathbb{K}^7$ equals $Y^* = \left\{ y \in \mathbb{K}^7 \mid y_1 = 0 \right\}$, thus $V = \left\{ y \in \mathbb{K}^7 \mid y_1 \neq 0 \right\}$ and $U = \Psi^{-1}(V) = \left\{ x \in \mathbb{K}^8 \mid x_1 \neq 0,\ x_5 \neq 0 \right\}$.
Obviously $\Psi(U) \neq \mathbb{K}^7$, so we are in case (ii) of the proof of Theorem 4 and find that $\rho := y_1$ vanishes on $\mathbb{K}^7 \setminus \Psi(U)$ and $\rho(\Psi(v)) \neq 0$. Thus $\psi := \rho \circ \Psi = x_1 x_5$ vanishes on $\mathbb{K}^8 \setminus U$ with $\psi(v) \neq 0$, and therefore

$$\mathbb{K}\left[x_1, \ldots, x_8\right]^G \left[\frac{1}{x_1x_5}\right] = \mathbb{K}\left[x_1x_5, x_2x_5, x_3x_5, x_4x_5, x_1x_6, x_1x_7, x_1x_8\right] \left[\frac{1}{x_1x_5}\right].$$

This localization works for the entire set $U_1 := U$. In the same way we find e.g. for $U_2 := \left\{ x \in \mathbb{K}^8 \mid x_2 \neq 0,\ x_6 \neq 0 \right\} \subseteq Z_8^*$ that a suitable localization is

$$\mathbb{K}\left[x_1, \ldots, x_8\right]^G \left[\frac{1}{x_2x_6}\right] = \mathbb{K}\left[x_1x_6, x_2x_6, x_3x_6, x_4x_6, x_2x_5, x_2x_7, x_2x_8\right] \left[\frac{1}{x_2x_6}\right].$$

4.3 Symmetric vector fields

To apply orbit space reduction to general G-symmetric polynomial vector fields, we also need an explicit representation of such vector fields. While $\mathcal{D}_G$ is clearly a module over the invariant algebra, the minimal number of module generators may be quite large, similar to the problem for $\mathbb{K}\,[x_1,\ldots,x_n]^G$. Making use of suitable localizations again, one finds that the number of generators required equals n (which is as small as possible in view of $Z_n^* \neq \emptyset$), provided that we impose one more condition on the group.

Theorem 5. *Let Q be the quotient field of $\mathbb{K}\,[x_1,\ldots,x_n]^G$ and q its transcendence degree over $\mathbb{K}$. Assume, in addition, that G is a subgroup of $SL(n,\mathbb{K})$, or that $Q = \mathbb{K}\,(x_1,\ldots,x_n)^G$. Moreover let $v \in Z_n^*$.*
Then there exist $\ell \in \{q, q+1\}$ and $\phi, \psi_1, \ldots, \psi_\ell \in \mathbb{K}\,[x_1,\ldots,x_n]^G$ with $\phi(v) \neq 0$ and $f_1,\ldots,f_n \in \mathcal{D}_G$ such that every $f \in \mathcal{D}_G$ can be written as

$$f = \mu_1 f_1 + \ldots + \mu_n f_n,$$

with $\mu_i \in \mathbb{K}[\psi_1,\ldots,\psi_\ell]\,[1/\phi]$, $1 \leq i \leq n$, and moreover

$$\mathbb{K}[x_1,\ldots,x_n]^G[1/\phi] = \mathbb{K}[\psi_1,\ldots,\psi_\ell]\,[1/\phi].$$

Proof. Let $v \in Z_n^*$, and $\psi, \psi_1, \ldots, \psi_\ell$ as in Theorem 4. Since $\epsilon_v(\mathcal{D}_G) = \mathbb{K}^n$, there exist $f_1,\ldots,f_n \in \mathcal{D}_G$ such that $f_1(v),\ldots,f_n(v)$ form a $\mathbb{K}$-basis of $\mathbb{K}^n$. Then the polynomial

$$\theta : x \mapsto \det(f_1(x),\ldots,f_n(x))$$

satisfies $\theta(v) \neq 0$, hence $\theta \neq 0$ and $f_1,\ldots,f_n$ form a $\mathbb{K}(x_1,\ldots,x_n)$-basis of $\mathbb{K}(x_1,\ldots,x_n)^n$. In particular any polynomial vector field f can be written as $f = \mu_1 f_1 + \ldots + \mu_n f_n$ with suitable $\mu_i \in \mathbb{K}\,(x_1,\ldots,x_n)$. If $f \in \mathcal{D}_G$, one verifies that the μ_i are G-invariant, and by Cramer's rule we have $\mu_i = \sigma_i/\theta$.

(i) In case $G \subseteq SL(n,\,\mathbb{K})$ one has

$$\begin{aligned} \theta(Tx) &= \det(f_1(Tx),\ldots,f_n(Tx)) \\ &= \det(Tf_1(x),\ldots,Tf_n(x)) \\ &= \det(T)\cdot\theta(x) = \theta(x) \end{aligned}$$

for all $T \in G$, and all $x \in \mathbb{K}^n$. Hence θ and every σ_i are G-invariant polynomials.
Therefore f is a linear combination of the f_i with coefficients in

$$\mathbb{K}[\psi_1,\ldots,\psi_\ell]\,[1/\psi]\,[1/\theta],$$

and $\psi \in \mathbb{K}[\psi_1, \ldots, \psi_\ell]$ by Remark 5. Moreover $\theta \in \mathbb{K}[x_1, \ldots, x_n]^G$, whence $\theta = \gamma/\psi^j$ with some $\gamma \in \mathbb{K}[\psi_1, \ldots, \psi_\ell]$ and some integer $j \geq 0$. Let $\phi := \gamma \cdot \psi$. Then $\phi(v) \neq 0$ and a standard argument (using that $\psi \in \mathbb{K}[\psi_1, \ldots, \psi_\ell]$) shows that

$$\mathbb{K}[\psi_1, \ldots, \psi_\ell]\,[1/\psi]\,[1/\theta] \subseteq \mathbb{K}[\psi_1, \ldots, \psi_\ell]\,[1/\phi].$$

(ii) If $Q = \mathbb{K}\left(x_1, \ldots, x_n\right)^G$ one can write $\mu_i = \widetilde{\sigma}_i/\widetilde{\theta}$ with G-invariant polynomials $\widetilde{\sigma}_i$ and $\widetilde{\theta}$, and then proceed as above.

□

Example 5. *Continuing Example 4, let*

$$G = \left\{ \begin{pmatrix} a \cdot E_4 & 0 \\ 0 & a^{-1} \cdot E_4 \end{pmatrix} ; a \in \mathbb{K}^* \right\} \subseteq SL(8, \mathbb{K})$$

and $v := (v_1, 0, 0, 0, v_5, 0, 0, 0) \in Z_8^*$, $v_1, v_5 \neq 0$. We have already seen that

$$\mathbb{K}\left[x_1, \ldots, x_8\right]^G \left[\frac{1}{x_1 x_5}\right] = \mathbb{K}\left[x_1x_5, x_2x_5, x_3x_5, x_4x_5, x_1x_6, x_1x_7, x_1x_8\right] \left[\frac{1}{x_1 x_5}\right],$$

with $x_1x_5(v) \neq 0$.

Denote by $e_i \in \mathbb{K}^8$ the standard basis vectors. Then

$$f_i := x_1 e_i,\ 1 \leq i \leq 4 \quad \textit{and} \quad f_i := x_5 e_i,\ 5 \leq i \leq 8$$

are G-symmetric vector fields such that $f_1(v), \ldots, f_8(v)$ span $\mathbb{K}^8$, and we find $\theta := det(f_1, \ldots, f_8) = x_1^4 x_5^4$, so we may choose $\phi = \psi$, using the notation in the proof of Theorem 5. Thus every $f \in \mathcal{D}_G$ can be written on U as

$$f = \mu_1 f_1 + \ldots + \mu_8 f_8 = (\mu_1 x_1, \mu_2 x_1, \mu_3 x_1, \mu_4 x_1, \mu_5 x_5, \mu_6 x_5, \mu_7 x_5, \mu_8 x_5),$$

$\mu_i \in \mathbb{K}\left[x_1x_5, x_2x_5, x_3x_5, x_4x_5, x_1x_6, x_1x_7, x_1x_8\right] \left[\frac{1}{x_1 x_5}\right].$

For the reduction of $\dot{x} = f(x)$, $f \in \mathcal{D}_G$, we may use the Hilbert map from Corollary 1, i.e.

$$\widehat{\widehat{\Psi}} := (x_1x_5, x_2x_5, x_3x_5, x_4x_5, x_1x_6, x_1x_7, x_1x_8)^{\mathrm{tr}}.$$

The reduced differential equation $\dot{y} = g(y)$ *is obtained from the identity* $g(\Psi(x)) = D\Psi(x) \cdot f(x)$*, thus*

$$g(y) = \begin{pmatrix} (\widetilde{\mu}_1 + \widetilde{\mu}_5)y_1 \\ \widetilde{\mu}_2 y_1 + \widetilde{\mu}_5 y_2 \\ \widetilde{\mu}_3 y_1 + \widetilde{\mu}_5 y_3 \\ \widetilde{\mu}_4 y_1 + \widetilde{\mu}_5 y_4 \\ \widetilde{\mu}_6 y_1 + \widetilde{\mu}_1 y_5 \\ \widetilde{\mu}_7 y_1 + \widetilde{\mu}_1 y_6 \\ \widetilde{\mu}_8 y_1 + \widetilde{\mu}_1 y_7 \end{pmatrix},$$

with $\widetilde{\mu}_i \in \mathbb{K}[y_1, \ldots, y_7]\left[\frac{1}{y_1}\right]$ *defined by* $\widetilde{\mu}_i \circ \Psi = \mu_i$*,* $1 \le i \le 8$*.*

Generally, it can be shown that for toral groups one always obtains a reduction map $\widehat{\widehat{\Psi}}$ to $\mathbb{K}^{n-s}$; hence the best possible scenario occurs.

4.4 Concluding remarks

- In the present note, our results were based on the assumption that $Z_n^* \neq \emptyset$, and we focussed on reduction near points of Z_n^*. We did not discuss the case when $Z_n^* = \emptyset$. In this case it is natural to consider the largest integer m such that $Z_m^* \neq 0$, and $v \in Z_m^*$. As noted above (see Theorem 3), then there exist common rational first integrals for the elements of $\mathcal{D}_G$ which (loosely speaking) make up for the deficiency in $\dim \epsilon_v(\mathcal{D}_G)$.

- Moreover we did not discuss reduction near $w \in Z_r$, with $r < m$ (including the case $m = n$). Here, some of the results and arguments from Sections 3 and 4 can be salvaged at least for reductive groups (in view of Remark 3), but matters become more complicated. We refer to (yet unpublished) work by Grosshans [9] on algebraic aspects.

- Finally one should not belittle the computational and algorithmic problems. While our approach does yield a reduction to a variety of codimension ≤ 2 in its ambient space, the ingredients may still be hard to obtain. This refers in particular to finding a primitive element as in the proof of Theorem 4. As it stands, our approach still requires knowledge of a generator set for $\mathbb{K}[x_1, \ldots, x_n]^G$. For toral groups, however, computations are quite straightforward.

- The main results (Theorems 4 and 5), as well as some facts mentioned here without proof, are taken from the first named author's upcoming doctoral thesis.

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