Algebraic Groups II

Summer School
Representations of Algebraic Groups and Lie Algebras in Characteristic p

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Some basic structure theory

K algebraically closed field, $\mathbb{G} \leq \mathrm{GL}_n(K)$ connected linear algebraic group

Definition

- $\mathbb{G}$ *almost simple* $\Leftrightarrow$ $\mathbb{G}$ has no non-trivial connected closed normal subgroup.
- $R(\mathbb{G})$ (the) maximal solvable connected normal subgroup, the *radical* of $\mathbb{G}$.
- $\mathbb{G}$ *semisimple* $\Leftrightarrow$ $R(\mathbb{G}) = \{1\}$.
- $R_u(\mathbb{G})$ (the) maximal unipotent connected normal subgroup, the *unipotent radical* of $\mathbb{G}$.
- $\mathbb{G}$ *reductive* $\Leftrightarrow$ $R_u(\mathbb{G}) = \{1\}$.

Remark

$\mathbb{G}$ almost simple $\Rightarrow$ $\mathbb{G}$ semisimple $\Rightarrow$ $\mathbb{G}$ reductive

Some basic structure theory

Theorem

- $R(\mathbb{G}) = \left(\bigcap_{B \leq \mathbb{G} \text{ Borel}} B\right)^o$, $R_u(\mathbb{G}) = \left(\bigcap_{B \leq \mathbb{G} \text{ Borel}} B_u\right)^o$
- $\mathbb{G}$ reductive $\Rightarrow R(\mathbb{G}) = Z(\mathbb{G})^o$, $\mathbb{G}'$ semisimple and $\mathbb{G} = Z(G)^o\mathbb{G}'$
- $\mathbb{G}$ semisimple $\Rightarrow \mathbb{G}$ has finitely many connected closed normal subgroups $H_1, \ldots, H_r$ and is the almost direct product of these. The H_i are called *almost simple components* of $\mathbb{G}$.

Example

- $R(\mathrm{GL}_n) \subset (\triangle) \cap (\nabla) = \mathbb{D}_n$ and normal $\Rightarrow R(\mathrm{GL}_n) = \mathrm{Z}(\mathrm{GL}_n) \cong \mathbb{G}_m$. $R_u(\mathrm{GL}_n) = \{1\} \Rightarrow \mathrm{GL}_n$ reductive (but not semisimple).
- Analogous: $R(\mathrm{SL}_n) = \{1\} \Rightarrow \mathrm{SL}_n$ semisimple
- $R((\triangle)) = (\triangle)$, $R_u((\triangle)) = \left\{\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\right\}$

The Lie-algebra

K algebraically closed field, $\mathbb{G} \leq \mathrm{GL}_n(K)$ linear algebraic group

Lie-algebra

Let $R = K[\epsilon]$ where $\epsilon^2 = 0$.

$$\mathrm{Lie}(\mathbb{G}) := \{A \in K^{n\times n} \mid I_n + \epsilon A \in \mathbb{G}(R)\},$$

the *Lie-algebra* of $\mathbb{G}$.

Remark

- $\mathrm{Lie}(\mathbb{G})$ with the matrix commutator is a Lie-algebra in the usual sense.
- $\dim(\mathbb{G}) = \dim_{\mathrm{K}}(\mathrm{Lie}(\mathbb{G}))$.

Example

- $\mathrm{Lie}(\mathrm{GL}_n) = K^{n\times n} = \mathfrak{gl}_n(K)$.
- $\mathrm{Lie}(\mathrm{SL}_n) = \{A \in K^{n\times n} \mid \mathrm{Tr}(A) = 0\} = \mathfrak{sl}_n(K)$
- $\mathrm{Lie}(O(F)) = \{A \in K^{n\times n} \mid AF + FA^{tr} = 0\} = \mathfrak{so}(F, K)$.

The Adjoint Representation and Root Systems

K algebraically closed field, $\mathbb{G} \leq \mathrm{GL}_n(K)$ connected reductive, $T \leq \mathbb{G}$ maximal torus

Definition

- The *Lie-linearization* (or *adjoint representation*) of $\mathbb{G}$ is:

$$\mathrm{Ad} : \mathbb{G} \to \mathrm{GL}(\mathrm{Lie}(\mathbb{G})),\ g \mapsto (A \mapsto g^{-1} A g).$$

- Set $X(T) := \mathrm{Mor}(T, \mathbb{G}_m)$ the *characters* of T.
- The non-trivial weights of T on $\mathrm{Lie}(\mathbb{G})$ via Ad

$$\Phi = \Phi(\mathbb{G}, T) = \{0 \neq \chi \in X(T) \mid \exists 0 \neq v \in \mathrm{Lie}(\mathbb{G}) :\ \forall\, t \in T : \mathrm{Ad}(t)v = \chi(t)v\}$$

  are called the *roots* of $\mathbb{G}$ (w.r.t. T).
- For $\alpha \in \Phi$ we call

$$\mathrm{Lie}(\mathbb{G})_\alpha = \{v \in \mathrm{Lie}(\mathbb{G}) \mid \forall t \in T : \mathrm{Ad}(t)v = \alpha(t)v\} \quad (\neq \{0\})$$

  the *root space* of α.

More on Root Systems

Theorem

- $\mathrm{Lie}(\mathbb{G}) = \mathrm{Lie}(T) \oplus \bigoplus_{\alpha\in\Phi} \mathrm{Lie}(\mathbb{G})_\alpha$
- $\dim_K(\mathrm{Lie}(\mathbb{G})_\alpha) = 1$ and $\mathrm{Lie}(\mathbb{G})_\alpha = \mathrm{Lie}(U_\alpha)$ for a unique unipotent T-invariant subgroup U_α (*one-parameter subgroup*).
- There are isomorphisms $u_\alpha : \mathbb{G}_a \to U_\alpha$ such that

$$t^{-1}u_\alpha(x)t = u_\alpha(\alpha(t)x).$$

- $\mathbb{G} = \langle T, U_\alpha \mid \alpha \in \Phi\rangle$
- $\mathbb{G}$ semisimple $\Rightarrow \mathbb{G} = \langle U_\alpha \mid \alpha \in \Phi\rangle$

Remark

- For $\chi \in X(T)$ we have $\chi \in \Phi \Leftrightarrow -\chi \in \Phi$.
- $B \leq \mathbb{G}$ Borel-subgroup yields a normalization on Φ via $\alpha > 0 \Leftrightarrow U_\alpha \leq B$. *Positive roots:* Φ^+.

The Weyl-Group

Definition

The *Weyl-group* of $\mathbb{G}$, $W = W(\mathbb{G}, T)$, is $W = N_{\mathbb{G}}(T)/T$.

Remark

- W is a finite Coxeter-group with set of reflections $\{s_\alpha \mid \alpha \in \Phi\}$ in bijection with Φ (s_α the *reflection corresponding to* α).
- W acts faithfully on $X(T)$ via $\chi^w(t) = \chi(t^{n^{-1}})$ for $t \in T, \chi \in X(T)$ and $w \in W$ with representative $n \in N_{\mathbb{G}}(T)$.
- W permutes Φ.
- $w = nN_{\mathbb{G}}(T) \in W$ then $n^{-1}U_\alpha n = U_{\alpha^w}$ for all $\alpha \in \Phi$.

Example

- $W(\mathrm{GL}_n) \cong S_n$ (Coxeter-type A_{n-1}).
- $W(\mathrm{SO}_{2n+1}) \cong C_2 \wr S_n$ (Coxeter-type B_n).

The Root Datum

Definition

- $X^\vee(T) := \mathrm{Mor}(\mathbb{G}_m, T)$, the *cocharacters* of T.
- Identify $\mathrm{Mor}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ and define $\langle -, - \rangle : X(T) \times X^\vee(T) \to \mathbb{Z}, (\chi, \mu) \mapsto \chi \circ \mu$.

Proposition

There are unique *coroots* $\alpha^\vee \in X^\vee(T), \ \alpha \in \Phi$ with:

- $\langle \alpha, \alpha^\vee \rangle = 2$.
- The action of s_α on $X(T)$ is given by $\chi \mapsto \chi - \langle \chi, \alpha^\vee \rangle \alpha$.
- $s_\alpha^\vee(\Phi^\vee) = \Phi^\vee$.

Remark

$(X(T), \Phi, X^\vee(T), \Phi^\vee)$ is an (abstract) root datum.

Theorem

The connected reductive groups are classified by their root data.

The Root Datum

Now let $\mathbb{G}$ be semisimple:

Definition

- $E := X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
- $Q(\Phi) := \langle \Phi \rangle$.
- $P(\Phi) := \{\lambda \in E \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \; \forall \, \alpha \in \Phi\}$.
- $\mathbb{G}$ *adjoint* $\Leftrightarrow Q(\Phi) = X(T)$.
- $\mathbb{G}$ *simply connected* $\Leftrightarrow P(\Phi) = X(T)$.

Remark

Φ is a root system. If we fix a system Φ^+ of positive roots we get a unique system Δ of simple roots.

- $|\Delta| = \dim(T)$.
- The Dynkin-diagram of $\mathbb{G}$ is the graph with vertices Δ, $|\langle \alpha, \beta^\vee \rangle| \cdot |\langle \beta, \alpha^\vee \rangle|$ edges between α and β and an arrow $\alpha \to \beta$ if $|\langle \alpha, \beta^\vee \rangle| > |\langle \beta, \alpha^\vee \rangle|$.

The Classification

Theorem

- The Dynkin-diagram of $\mathbb{G}$ is connected iff $\mathbb{G}$ is almost simple.
- $\mathbb{G}$ semisimple $\Rightarrow$ The connected components of the Dynkin-diagram correspond to the almost simple components of $\mathbb{G}$.
- The connected almost simple linear algebraic groups are classified by their Dynkin-diagrams and the position of $X(T)$ in the chain $Q(\Phi) \subset X(T) \subset P(\Phi)$.

The connected almost simple linear algebraic groups

Coxeter-type	s.c.	intermed. forms	adj.	$[P(\Phi) : Q(\Phi)]$
$A_n,\ n \geq 1$	SL_{n+1}	...	PSL_{n+1}	# div. of $n+1$
$B_n,\ n \geq 2$	Spin_{2n+1}	-	SO_{2n+1}	2
$C_n,\ n \geq 3$	Sp_{2n}	-	PGSp_{2n}	2
$D_n,\ n \geq 4$	Spin_{2n+1}	SO_{2n} (+2 if n even)	PCSO_{2n}	4
$E_n,\ n = 6, 7, 8$	$(E_n)_{sc}$	-	$(E_n)_{ad}$	$3, 2, 1$
F_4		F_4		1
G_2		G_2		1