In his seminal paper of 1771, Lagrange found that certain polynomials of degree 6 called resolvents could be used to determine whether a quintic polynomial was solvable in radicals. Among the various resolvents later discovered, the Perrin-McClintock resolvent has some particularly noteworthy properties. We shall discuss these properties and their application to solvable quintics. The properties suggest that the Perrin-McClintock resolvent may be unique. We discuss this question and relate it to the representation theory of the general linear group, especially zero weight spaces and plethysms of a special form.
Properties of the Perrin-McClintock Resolvent

Property 1: a polynomial function

\[ d = 1, 2, \ldots \]

\[ f(x, y) = \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i, \quad a_i \in \mathbb{C} \]

\[ = a_0 x^d + \binom{d}{1} a_1 x^{d-1} y + \binom{d}{2} a_2 x^{d-2} y^2 + \ldots + \binom{d}{d} a_d y^d \]

\[ \leftrightarrow (a_0, a_1, \ldots, a_d) \]

\[ V_d \] is vector space over \( \mathbb{C} \) spanned by all such \( f(x, y) \)

\[ \mathbb{A}^2 = \mathbb{C}^2 = \left\{ \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \right\} \]

The Perrin-McClintock resolvent

a polynomial, \( K : V_5 \times \mathbb{A}^2 \to \mathbb{C} \)

\[ K(a_0, a_1, a_2, a_3, a_4, a_5; x, y) = \sum_{j=0}^{6} k_j(a_0, a_1, a_2, a_3, a_4, a_5) x^{6-j} y^j \]

\[ R_f(x) = K(f, \begin{pmatrix} x \\ 1 \end{pmatrix}) \]
Example 1: \( f(x) = x^5 + 10a_2x^3 + 5a_4x + a_5 \) with \( a_4 = 4a_2^2 \)
\[ K(f, v) = (3a_2^6 + a_2a_3^2)x^6 - 125a_2^4a_5x^5y + (4080a_2^7 - 15a_2^2a_3^2)x^4y^2 \]
\[ + 1000a_2^5a_5x^3y^3 + (960a_2^8 + 70a_2^3a_3^2)x^2y^4 + (128a_2^9a_5 + a_2a_3^3)xy^5 \]

Example 2: \( f(x) = x^5 + 5x^4 + 9x^3 + 5x^2 - 4x - 5 \)
\[ K(f, v) = \frac{1}{80000}(-498x^6 - 5900x^5y - 22662x^4y^2 - 41320x^3y^3 - 36254x^2y^4 \]
\[ -6860xy^5 + 8150y^6) \]

Example 3: \( f(x) = x^5 - 8x^4 + 5x^3 - 6x^2 + 8x - 4 \)
\[ K(f, v) = -5681513x^6 + 22679884x^5y - 42714844x^4y^2 + 6325088x^3y^3 \]
\[ + 16299792x^2y^4 - 18575936xy^5 + 5294016y^6 \]
Properties of the Perrin-McClintock Resolvent

Property 2: a covariant

\( SL(2, \mathbb{C}), g = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\} \)

action on \( V_d \)
\( g \cdot x = dx - by \)
\( g \cdot y = -cx + ay \)

\[
\begin{align*}
  f &= \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i \\
  \Rightarrow g \cdot f &= \sum_{i=0}^{d} \binom{d}{i} a_i (dx - by)^{d-i} (-cx + ay)^i
\end{align*}
\]

action on \( \mathbb{A}^2 \)

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} a\lambda + b\mu \\ c\lambda + d\mu \end{pmatrix}
\]

The covariant property

1. \( K(g \cdot f, g \cdot v) = K(f, v) \) for all \( g \in SL(2, \mathbb{C}), f \in V_d, v \in \mathbb{A}^2 \)

2. coefficients of \( x \) and \( y \) terms form irreducible representation of \( SL(2, \mathbb{C}) \)

3. source of covariant is \( K(f, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \)

completely determines \( K \)
Properties of the Perrin-McClintock Resolvent

The Hessian cubic covariant

\[ g \in SL(2, \mathbb{C}), g = \begin{pmatrix} 5 & 2 \\ 17 & 7 \end{pmatrix} \]

action on \( V_3 \)

\[ g \cdot x = 7x - 2y \]
\[ g \cdot y = -17x + 5y \]

\[ f = a_0 x^3 + 3a_1 x^2 y + 3a_2 xy^2 + a_3 y^3 \]
\[ g \cdot f = a_0(7x - 2y)^3 + 3a_1(7x - 2y)^2(-17x + 5y) + 3a_2(7x - 2y)(-17x + 5y)^2 + a_3(-17x + 5y)^3 \]

action on \( \mathbb{A}^2 \)

\[ \begin{pmatrix} 5 & 2 \\ 17 & 7 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 5\lambda + 2\mu \\ 17\lambda + 7\mu \end{pmatrix} \]

\[ H(a_0, a_1, a_2, a_3; x, y) = \frac{1}{36} \text{Det} \left( \begin{array}{ccc} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x^2} \end{array} \right) \]
\[ = (a_0a_2 - a_1^2)x^2 + (a_0a_3 - a_1a_2)xy + (a_1a_3 - a_2^2)y^2 \]
\[ g = \begin{pmatrix} 5 & 2 \\ 17 & 7 \end{pmatrix} \]

\[ f = 4x^3 + 3 \times 5x^2y + 3 \times (-6)xy^2 + (-1)y^3 \]

\[ g \cdot f = -42624x^3 + 37128x^2y - 10779xy^2 + 1043y^3 \]

\[ v = \begin{pmatrix} 8 \\ 3 \end{pmatrix}; \quad g \cdot v = \begin{pmatrix} 46 \\ 157 \end{pmatrix} \]

The covariant property

1. \( H(g \cdot f, g \cdot v) = H(f, v) \) for all \( g \in G, \ f \in V_d, \ v \in \mathbb{A}^2 \)

\[ H(f, v) = H(4, 5, -6, -1; 8, 3) = -2881 \]

\[ H(g \cdot f, g \cdot v) = H(-42624, 12376, -3593, 1043; 46, 157) \]

\[ = -2881 \]

2. coefficients of \( x \) and \( y \) terms form irreducible representation of \( SL(2, \mathbb{C}) \)

\( (a_0a_2 - a_1^2), (a_0a_3 - a_1a_2), (a_1a_3 - a_2^2) \)

3. source of covariant is \( H(f, \begin{pmatrix} 1 \\ 0 \end{pmatrix} ) = a_0a_2 - a_1^2 \)

completely determines \( H \)

algebraic meaning: \( H(f, v) = 0 \) for all \( v \in \mathbb{A}^2 \) if and only if there is a linear form, say \( g = ax + by, \) such that \( f = g^3. \)

For example, \( f(x, y) = 64x^3 - 144x^2y + 108xy^2 - 27y^3. \)

\[ H(f, v) \equiv 0, \ f(x, y) = (4x - 3y)^3 \]

[Abdesselam and Chipalkatti]
Properties of the Perrin-McClintock Resolvent

Property 3: solvable quintics

Theorem. Let \( f(x) = a_0x^5 + 5a_1x^4 + 10a_2x^3 + 10a_3x^2 + 5a_4x + a_5 \) be an irreducible quintic polynomial in \( \mathbb{Q}[x] \). Then \( f(x) \) is solvable in radicals if and only if \( R_f(x) \) has a rational root or is of degree 5.

Example 2: \( f(x) = x^5 + 5x^4 + 9x^3 + 5x^2 - 4x - 5 \)

\[
K(f, v) = \frac{1}{8000}(-498x^6 - 5900x^5y - 22662x^4y^2 - 41320x^3y^3 - 36254x^2y^4
-6860xy^5 + 8150y^6)
\]

\[
R_f(x) = \frac{1}{8000}(-498x^6 - 5900x^5 - 22662x^4 - 41320x^3 - 36254x^2
-6860x + 8150)
\]

has root 1/3. Hence, \( f(x) \) is solvable in radicals.

Example 3: \( f(x) = x^5 - 8x^4 + 5x^3 - 6x^2 + 8x - 4 \)

\[
K(f, v) = -5681513x^6 + 22679884x^5y - 42714844x^4y^2 + 6325088x^3y^3 +
16299792x^2y^4 - 18575936xy^5 + 5294016y^6
\]

\[
R_f(x) = -5681513x^6 + 22679884x^5 - 42714844x^4 + 6325088x^3 +
16299792x^2 - 18575936x + 5294016
\]

does not have a rational root. Hence, \( f(x) \) is not solvable in radicals.

Get elegant way to find solutions in radicals

Cayley to McClintock (McClintock, p.163): "McClintock completes in a very elegant manner the determination of the roots of the quintic equation . . . ."
Properties of the Perrin-McClintock Resolvent

Property 4: global information

**Problem:** use resolvents to obtain global information about solvable quintics

**Example 1** (Perrin):

\[ f(x) = a_0 x^5 + 10a_2 x^3 + 5a_4 x + a_5 \]

with \( a_4 = 4a_2^2 \)

\[ R_f(x) = (3a_2^6 + a_2a_5^2)x^6 - 125a_4^4a_5 x^5 + (4080a_2^7 - 15a_2^2a_5^2)x^4 \]

\[ + 1000a_2^5a_5x^3 + (960a_2^8 + 70a_2^3a_5^2)x^2 + (128a_2^6a_5 + a_2a_5^3)x \]

has root 0. Hence, \( f(x) \) is solvable in radicals.

**Example 2:** the McClintock parametrization

Have mapping \( \varphi \), a rational function,

\[ \varphi : A_Q^4 \rightarrow A_Q^4 \]

\[ (p, r, w, t) \rightarrow (\gamma, \delta, \varepsilon, \zeta) \]

\( (\gamma, \delta, \varepsilon, \zeta) \) identified with \( f(x) = x^5 + 10\gamma x^3 + 10\delta x^2 + 5\varepsilon x + \zeta \)

The polynomial \( f(x) \) is solvable (its resolvent \( R_f(x) \) has \( t \) as a root).

inverse map exists, rational function

need \( R_f(x) \) to have rational root \( t \)

difficulty: if quintic factors, \( t \) may be complex or irrational real

so don’t quite parametrize all solvable quintics

8
**Example 3:** Brioschi quintics [Elia]

\[ f(x) = x^5 - 10zx^3 + 45z^2x - z^2 \]

\[ R_f(x) = (-z^5 + 128z^6)x^6 + 400z^6x^5 + (-15z^6 - 46080z^7)x^4 \]
\[ + 40000z^7x^3 + (-95z^7 - 51840z^8)x^2 \]
\[ + (z^7 + 1872z^8)x - 25z^8 \]

If \( z \) is a non-zero integer, then \( f(x) \) is solvable in radicals.

**Example 4:** subject to certain explicitly defined polynomials not vanishing, if \( f_0 \) is an irreducible quintic such that \( R_{f_0} \) has a root \( t_0 \in \mathbb{R} \), then every Euclidean open neighborhood of \( f_0 \) contains a solvable quintic.
Dickson’s Factorization

Action of $S_5$

$S_5$: symmetric group on 5 letters

Action of $S_5$ on polynomials

\[ f(x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}[x_1, x_2, x_3, x_4, x_5] \]

\[ \sigma \in S_5 \]

\[ \sigma \cdot f = f(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3}, x_{\sigma 4}, x_{\sigma 5}) \]

Example

\[ f = x_1 x_2 - x_1 x_3 + x_2 x_3 - x_1 x_4 - x_2 x_4 + x_3 x_4 + x_1 x_5 - x_2 x_5 - x_3 x_5 + x_4 x_5 \]

\[ \sigma = (132) \]

\[ \sigma \cdot f = x_3 x_1 - x_3 x_2 + x_1 x_2 - x_3 x_4 - x_1 x_4 + x_2 x_4 + x_3 x_5 - x_1 x_5 - x_2 x_5 + x_4 x_5 \]
Dickson’s Factorization

The group $F_{20}$

$S_5$: symmetric group on 5 letters

$F_{20}$: subgroup of $S_5$ generated by $(12345)$ and $(2354)$

$S_5 = \bigcup_{i=1}^{6} \tau_i F_{20}$

$\tau_1 = (1), \tau_2 = (12), \tau_3 = (13), \tau_4 = (23), \tau_5 = (123), \tau_6 = (132)$

**Theorem.** Let $f(x) \in \mathbb{Q}[x]$ be an irreducible quintic. Then $f(x)$ is solvable in radicals if and only if its Galois group is conjugate to a subgroup of $F_{20}$.

Problem: extend resolvent program to polynomials of higher degree. (What replaces $F_{20}$?)
Dickson’s Factorization

Malfatti’s resolvent

\[ \Phi(x_1, x_2, x_3, x_4, x_5) = x_1x_2 - x_1x_3 + x_2x_3 - x_1x_4 - x_2x_4 + x_3x_4 + x_1x_5 - x_2x_5 - x_3x_5 + x_4x_5 \]
\[ = (x_1 - x_5)(x_2 - x_5) + (x_2 - x_5)(x_3 - x_5) + (x_3 - x_5)(x_4 - x_5) \]
\[ - (x_2 - x_5)(x_4 - x_5) - (x_4 - x_5)(x_1 - x_5) - (x_1 - x_5)(x_3 - x_5) \]

Properties:

(1) homogeneous of degree 2 in \( x_1, x_2, x_3, x_4, x_5 \)

(2) for any \( \beta \in \mathbb{C} \), \( \Phi(x_1 + \beta, x_2 + \beta, x_3 + \beta, x_4 + \beta, x_5 + \beta) = \Phi(x_1, x_2, x_3, x_4, x_5) \)

(3) highest power to which any \( x_i \) appears is 1

(4) \( (12345)\Phi = \Phi \)
\[ (2354)\Phi = -\Phi \]

Note: Malfatti resolvent, put \( \Phi^2 = \Theta \)

\[ R(x) = (x - \Theta)(x - \tau_2\Theta)(x - \tau_3\Theta)(x - \tau_4\Theta)(x - \tau_5\Theta)(x - \tau_6\Theta) \]

polynomial in \( a'_i \)’s
rational root if and only if \( f(x) \) solvable in radicals
for resolvents of this form, lowest possible degree in \( \Theta \)
rediscovered by Jacobi (1835), Cayley (1861), Dummit (1991)
Dickson’s Factorization

Roots of the resolvent

\[ S_5 = \bigcup_{i=1}^{6} \tau_i F_{20} \]

\[ \tau_1 = (1), \tau_2 = (12), \tau_3 = (13), \tau_4 = (23), \tau_5 = (123), \tau_6 = (132) \]

The Malfatti resolvent

\[ \Phi(x_1, x_2, x_3, x_4, x_5) = x_1x_2 - x_1x_3 + x_2x_3 - x_1x_4 - x_2x_4 + x_3x_4 + x_1x_5 - x_2x_5 - x_3x_5 + x_4x_5 \]

\[ = (x_1 - x_5)(x_2 - x_5) + (x_2 - x_5)(x_3 - x_5) + (x_3 - x_5)(x_4 - x_5) \]

\[ - (x_2 - x_5)(x_4 - x_5) - (x_4 - x_5)(x_1 - x_5) - (x_1 - x_5)(x_3 - x_5) \]

\[ \Psi(x_1, x_2, x_3, x_4, x_5) = (x_1x_2x_3x_4x_5)\Phi(1/x_1, 1/x_2, 1/x_3, 1/x_4, 1/x_5) \]

where does \( \Psi \) come from?

need highest power to which a root appears in \( \Phi \) is \( \leq 1 \)

homogeneous of degree 3 in \( x_1, x_2, x_3, x_4, x_5 \)

Perrin-McClintock resolvent: for \( i = 1, 2, 3, 4, 5, 6 \), put \( \Phi_i = \tau_i \Phi, \Psi_i = \tau_i \Psi \)

\[ K(f; v) = a_0^6 \prod_{i=1}^{6}((\tau_i \Phi)x - (\tau_i \Psi)y) \]

\[ = a_0^6(\prod_{i=1}^{6}(\tau_i \Phi)) \prod_{i=1}^{6}(x - ((\tau_i \Psi)/(\tau_i \Phi))y) \]
Constructing resolvents

Setting I: covariants

Find covariants of the form

\[ K(f; v) = a_0^m \prod_{i=1}^{6} ((\tau_i \Phi)x - (\tau_i \Psi)y) \]

with

(1) \( \Phi(x_1, x_2, x_3, x_4, x_5) \) homogeneous of degree \( w \equiv 2 \) (mod 5) in \( x_1, x_2, x_3, x_4, x_5 \)

(2) highest power to which a root appears in \( \Phi \) is \( \leq d = \frac{2w+1}{5} \)

(3) for any \( \beta \in \mathbb{C}, \Phi(x_1+\beta, x_2+\beta, x_3+\beta, x_4+\beta, x_5+\beta) = \Phi(x_1, x_2, x_3, x_4, x_5) \)

(4) \( (12345) \Phi = \Phi \)
\( (2354) \Phi = -\Phi \)

\[ \Psi(x_1, x_2, x_3, x_4, x_5) = (x_1x_2x_3x_4x_5)^d \Phi(-1/x_1, -1/x_2, -1/x_3, -1/x_4, -1/x_5) \]

Recall: source determines covariant.

source is \( a_0^m \prod_{i=1}^{6} ((\tau_i \Phi) \)
Constructing resolvents

Setting II: polynomials in roots

For \( w \equiv 2 \pmod{5} \), put \( d = \frac{2w+1}{5} \). Find \( \Phi(x_1, x_2, x_3, x_4, x_5) \)

1. homogeneous of degree \( w \) in \( x_1, x_2, x_3, x_4, x_5 \)
2. the highest power to which any \( x_i \) appears in \( \Phi \) is \( \leq d \)
3. for any \( \beta \in \mathbb{C} \), \( \Phi(x_1+\beta, x_2+\beta, x_3+\beta, x_4+\beta, x_5+\beta) = \Phi(x_1, x_2, x_3, x_4, x_5) \)
4. \( (12345)\Phi = \Phi \)
   \( (2354)\Phi = -\Phi \)

For covariant, need \( w \equiv 2 \pmod{5} \)

Malfatti is only such polynomial of degree 2
Problems.

1. no \( SL_2(\mathbb{C}) \) action

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} x_i = (dx_i - b)/(cx_i + a)
\]

\[
\begin{pmatrix}
  0 & 1 \\
  -1 & 0
\end{pmatrix} x_i = -1/x_i
\]

2. the highest power to which any \( x_i \) appears in \( \Phi \) is \( \leq d \)

\[ f(x_1, \ldots, x_p) \in \mathbb{Z}[x_1, \ldots, x_p], \ f = \sum \lambda(a)x_1^{a_1} \ldots x_p^{a_p} \]

\[ R(f) = \max\{a_i : 1 \leq i \leq p, \lambda(a) \neq 0\} \]

If \( f(x_1, \ldots, x_p) \) is symmetric in \( x_1, \ldots, x_p \), then \( f(x_1, \ldots, x_p) = \sum \tau(b)\sigma_1^{b_1} \ldots \sigma_p^{b_p} \)

\[ \sigma_i = \text{ith elementary symmetric function.} \]

\[ D(f) = \max\{b_1 + \ldots + b_p : \tau(b) \neq 0\} \]

**Theorem.** If \( f(x_1, \ldots, x_p) \) is symmetric in \( x_1, \ldots, x_p \), then \( R(f) = D(f) \).
Constructing resolvents

Setting III: matrix variables

Translation:

For \( d \equiv 1(\text{mod} \ 2) \), find matrix polynomials \( \tilde{F} \left( \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & \bar{y}_4 & \bar{y}_5 \end{pmatrix} \right) \).

(1) \( \tilde{F} \) is homogenous of degree \( d \) in each column, i.e.,

\[
\tilde{F} = \sum c(e) \bar{x}_1^{e_1} \bar{y}_1^{d-e_1} \cdots \bar{x}_5^{e_5} \bar{y}_5^{d-e_5}
\]

(2) \( \tilde{F} \) is left \( U \)-invariant, i.e.,

\[
\tilde{F} \left( \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & \bar{y}_4 & \bar{y}_5 \end{pmatrix} \right) = \tilde{F} \left( \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & \bar{y}_4 & \bar{y}_5 \end{pmatrix} \right)
\]

or

\[
\sum c(e) (\bar{x}_1 + \beta \bar{y}_1)^{e_1} \bar{y}_1^{d-e_1} \cdots (\bar{x}_5 + \beta \bar{y}_5)^{e_5} \bar{y}_5^{d-e_5} = \sum c(e) \bar{x}_1^{e_1} \bar{y}_1^{d-e_1} \cdots \bar{x}_5^{e_5} \bar{y}_5^{d-e_5}
\]
(3) \( \tilde{F} \) has left \( T \)-weight 1, i.e., 
\[ 5d - 2(e_1 + e_2 + e_4 + e_4 + e_5) = 1 \]

or

\[
\tilde{F} \left( \begin{pmatrix} \lambda & 0 & 1/\lambda \\ 0 & 1/\lambda \end{pmatrix} \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & \bar{y}_4 & \bar{y}_5 \end{pmatrix} \right) = \frac{1}{\lambda} \tilde{F} \left( \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & \bar{y}_4 & \bar{y}_5 \end{pmatrix} \right)
\]

or

\[
\sum c_{(e)} (\lambda \bar{x}_1 + \frac{1}{\lambda} \bar{y}_1)^{e_1} \bar{y}_1^{d-e_1} \cdots (\lambda \bar{x}_5 + \frac{1}{\lambda} \bar{y}_5)^{e_5} \bar{y}_5^{d-e_5} = \frac{1}{\lambda} \sum c_{(e)} \bar{x}_1^{e_1} \bar{y}_1^{d-e_1} \cdots \bar{x}_5^{e_5} \bar{y}_5^{d-e_5}
\]

(4) \( S_5 \) acts on vector variables by permuting columns

\[
(12345) \bar{F} = \bar{F} \\
(2354) \bar{F} = -\bar{F}
\]

\[
\bar{F} \left( \begin{pmatrix} \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \\ \bar{y}_2 & \bar{y}_3 & \bar{y}_4 & \bar{y}_5 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix} \right) = \bar{F} \left( \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & \bar{y}_4 & \bar{y}_5 \end{pmatrix} \right)
\]

\[
\bar{F} \left( \begin{pmatrix} \bar{x}_1 & \bar{x}_3 & \bar{x}_5 \\ \bar{y}_1 & \bar{y}_3 & \bar{y}_5 \end{pmatrix} \begin{pmatrix} \bar{x}_2 & \bar{x}_4 \\ \bar{y}_2 & \bar{y}_4 \end{pmatrix} \right) = -\bar{F} \left( \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & \bar{y}_4 & \bar{y}_5 \end{pmatrix} \right)
\]
Constructing resolvents

Setting I: find covariants of the form $K(f; v) = a_0^n \prod_{i=1}^{6} ((\tau_i \Phi)x - (\tau_i \Psi)y)$

Setting II: find $\Phi(x_1, x_2, x_3, x_4, x_5)$

Setting III: find matrix polynomials $\tilde{F}$

**Definition.** For $i = 1, 2$, let $K_i : V_5 \times \mathbb{A}^2 \to \mathbb{C}$ be covariants as in Setting I. Say $K_1 \sim K_2$ if and only if there are $\bar{\mu}, \bar{\rho} \in \mathbb{C}[V_5]^{SL_2(\mathbb{C})}$ with $\bar{\mu}K_1(x, y) = \bar{\rho}K_2(x, y)$.

Let $\Phi$ and $\Phi'$ be as in Setting II. Say $\Phi \sim \Phi'$ if and only if \( \frac{\psi}{\Phi} = \frac{\psi'}{\Phi'} \).

Setting I and Setting II:

$K = a_0^n \prod_{i=1}^{6} ((\tau_i \Phi)x - (\tau_i \Psi)y), K' = a_0^n \prod_{i=1}^{6} ((\tau_i \Phi)x - (\tau_i \Psi)y)$

$K \sim K'$ if and only if $\Phi \sim \Phi'$

Setting II and Setting III: there is vector space isomorphism between

- $\Phi$ homogeneous of degree $w$
- $\tilde{F}$ homogeneous of degree $2w + 1$

also, have algebra homomorphism

have mapping $\Omega : \mathbb{C} \left[ \begin{array}{c} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{x}_2 \\ \tilde{y}_2 \\ \tilde{x}_3 \\ \tilde{y}_3 \\ \tilde{x}_4 \\ \tilde{y}_4 \\ \tilde{x}_5 \\ \tilde{y}_5 \end{array} \right] \to \mathbb{C}[x_1, x_2, x_4, x_4, x_5]$

\( \tilde{x}_i \to x_i, \tilde{y}_i \to 1 \)
Constructing resolvents

finitely generated modules

\[ R_w \subset \mathbb{C}[x_1, x_2, x_3, x_4, x_5] \]

For \( w \equiv 0 \pmod{5} \), \( w \geq 0 \), \( R_w \) is vector space spanned by all linear combinations \( \Phi \) of products \( (x_{i_1} - x_{j_1}) \cdots (x_{i_w} - x_{j_w}) \) such that

1. each \( x_i \) appears \( \frac{2w}{5} \) times in every product
2. \( (12345)\Phi = \Phi \), \( (2354)\Phi = \Phi \)

\[ R = \bigoplus R_w \]

\[ M_w \subset \mathbb{C}[x_1, x_2, x_3, x_4, x_5] \]

For \( w \equiv 2 \pmod{5} \), \( w \geq 0 \), \( M_w \) is vector space spanned by \( \Phi(x_1, x_2, x_3, x_4, x_5) \)

1. homogeneous of degree \( w \) in \( x_1, x_2, x_3, x_4, x_5 \)
2. the highest power to which any \( x_i \) appears in \( \Phi \) is \( \leq \frac{2w+1}{5} \)
3. for any \( \beta \in \mathbb{C}, \Phi(x_1+\beta, x_2+\beta, x_3+\beta, x_4+\beta, x_5+\beta) = \Phi(x_1, x_2, x_3, x_4, x_5) \)
4. \( (12345)\Phi = \Phi \), \( (2354)\Phi = -\Phi \)

\[ M = \bigoplus M_w \]
Theorem. (a) $R$ is finitely generated $\mathbb{C}$-algebra.

(b) $\Delta = \Phi_2 \Psi_7 - \Phi_7 \Psi_2 \neq 0$ and is in $R$.

(c) $\Phi \in M, \Phi = \frac{r_1}{\Delta} \Phi_2 + \frac{r_2}{\Delta} \Phi_7$

(d) $M$ is finitely generated $R$-module

(e) $\dim_{Q(R)} M \otimes_R Q(R) = 2$
Poincaré series

Hilbert - Serre theorem

Recall $R = \bigoplus R_w$, is finitely generated $\mathbb{C}$-algebra

$M = \bigoplus M_w$, $M_w$ polynomials as in Setting II
is finitely generated $R$-module

Poincaré series: $P(M, t) = \sum_{w \equiv 2 \, (\text{mod } 5)} \dim M_w$

**Theorem** (Hilbert, Serre, applied here). Let $\gamma$ be the number of generators of $R$. Then

$$P(M, t) = \frac{f(t)}{\prod_{i=1}^{\gamma} (1 - t^{d_i})}$$

for suitable positive integers $d_i$ and $f(t) \in \mathbb{Z}[t]$.

Problem: determine $P(M, t)$.

Determine $\dim M_w$.
Poincaré series

\( GL_m - GL_n \) duality

to understand:

(2) \( \bar{F} \) is left \( U \)-invariant

(3) \( \bar{F} \) has left \( T \)-weight 1

\( T_r \subset GL_r \); subgroup consisting of diagonal matrices

\( U_r \subset GL_r \); subgroup consisting of upper triangular matrices, 1’s on diagonal

A highest weight of an irreducible polynomial representation of \( GL_r \) with respect to the Borel subgroup \( T_r U_r \) is a character of the form \( \chi = e_1 \chi_1 + \cdots + e_r \chi_r \) where \( e_1 \geq \ldots \geq e_r \geq 0 \). If \( e_\ell \) is the last non-zero \( e_i \), we say that the highest weight \( \chi \) has depth \( \ell \).

**Theorem** (\( GL_m - GL_n \) duality) [Howe, Section 2.1.2]. Let \( U \) and \( V \) be finite-dimensional vector spaces over \( \mathbb{C} \). The symmetric algebra \( S(U \otimes V) \) is multiplicity-free as a \( GL(U) \times GL(V) \) module. Precisely, we have a decomposition

\[
S(U \otimes V) = \sum_D \rho^D_U \otimes \rho^D_V
\]

of \( GL(U) \times GL(V) \)-modules. Here \( D \) varies over all highest weights of depth at most \( \min\{\dim U, \dim V\} \).
Translation

$M_{2,5}$: the algebra consisting of all $2 \times 5$ matrices with entries in $\mathbb{C}$.

$GL_2$ acts on $M_{2,5}$ by left multiplication: $g \cdot m = gm$ for all $g \in GL_2$ and $m \in M_{2,5}$.

$GL_5$ acts on $M_{2,5}$ by right multiplication: $g \cdot m = mg^{-1}$ for all $g \in GL_5$ and $m \in M_{2,5}$.

These actions commute and give an action of $G = GL_2 \times GL_5$ on $M_{2,5}$ and $\mathbb{C}[M_{2,5}]$.

$$M_{2,5} \hookrightarrow \mathbb{A}^2 \otimes (\mathbb{A}^5)^*$$

$$\mathbb{C}[M_{2,5}] \hookrightarrow S((\mathbb{A}^2)^* \otimes \mathbb{A}^5)$$

Suppose that $d \equiv 1 \pmod{2}$, $5d = 2w + 1$ and that

(2) $\widetilde{F}$ is left $U$-invariant

(3) $\widetilde{F}$ has left $T$-weight 1

then: the terms $\widetilde{F} = \tilde{v} \otimes V_D$ appear when

$\tilde{v}$: highest weight vector of irreducible representation $GL_2$, highest weight $(w + 1)\chi_1 + w\chi_2$

$\rho_D^\chi$ is irreducible representation of $GL_5$, highest weight $(w + 1)\chi_1 + w\chi_2$

Note. can explicitly construct the invariants $\widetilde{F}$ in terms of determinants using Young diagrams and straightening [Pommerening].
Poincaré series

Zero weight space

to understand:

(1) $\tilde{F}$ is homogenous of degree $d$ in each column

recall: $w \equiv 2(\text{mod } 5)$, $5d = 2w + 1$

$$T_5 = \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix} \right\}, \ U_5 : \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 1 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 1 & a_{34} & a_{35} \\ 0 & 0 & 0 & 1 & a_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

$\rho : GL_5 \to GL(V)$

$V_0 = \{ v \in V : \rho(t)v = (a_1a_2a_3a_4a_5)^e v \} = 0$ weight space of $V$

translation: $\tilde{F} \in \mathbb{C}[M_{2,5}], \ t \in T_5, \ m = (v_1, \ldots, v_5) \in M_{2,5}$

$$(t\tilde{F})(v_1, \ldots, v_5) = \tilde{F}((v_1, \ldots, v_5)t) = \tilde{F}(a_1v_1, \ldots, a_5v_5) = (a_1a_2a_3a_4a_5)^d \tilde{F}(v_1, \ldots, v_5)$$
Proposition. Let $w \equiv 2(\text{mod } 5)$ and $d = \frac{2w+1}{5}$. Let $\rho : GL_5 \to GL(V)$ be the irreducible representation having highest weight $(w+1)\chi_1 + w\chi_2$. The vector space consisting of all $\bar{F} \in \mathbb{C}[M_{2,5}]$ such that

1. $\bar{F}$ is homogenous of degree $d$ in each column,
2. $\bar{F}$ is left $U$-invariant,
3. $\bar{F}$ has left $T$-weight 1

is isomorphic to the 0-weight space of $V$. 
Poincaré series

$S_5$ action on zero weight space

to understand:

$$(4) \quad (12345)\bar{F} = \bar{F}, \quad (2354)\bar{F} = -\bar{F}$$

$S_5$ acts on 0–weight space, $V_0$

$$V_0 = \bigoplus m_\chi V_\chi$$

$V_\chi$ runs over all irreducible representations of $S_5$

$m_\chi$ is multiplicity with which $V_\chi$ appears in $V_0$

$S_5$ has 7 irreducible representations

$[5], [41], [32], [31^2], [2^21], [21^3], [1^5]$ 

$\overline{\rho} : F_{20} \rightarrow \{ \pm 1 \}$

$$\overline{\rho}(12345) = 1$$

$$\overline{\rho}(2354) = -1.$$ 

$\overline{\rho}$ appears with multiplicity 1 in both $[3 \ 2]$ and $[1^5]$. It does not appear in any of the other 5 irreducible representations.
Proposition. Let \( w \equiv 2 \pmod{5} \) and \( d = \frac{2w+1}{5} \). Let \( \rho : GL_5 \to GL(V) \) be the irreducible representation having highest weight \((w + 1)\chi_1 + w\chi_2\). The vector space consisting of all \( F \in \mathbb{C}[M_{2,5}] \) such that

1. \( F \) is homogenous of degree \( d \) in each column,
2. \( F \) is left \( U \)-invariant,
3. \( F \) has left \( T \)-weight 1,
4. \((12345)F = F \) and \((2354)F = -F\)

is isomorphic to the vector space consisting of vectors \( v \) in the 0-weight space of \( V \) which satisfy \((12345)v = v \) and \((2354)v = -v\).

The dimension of this vector space is the sum of the multiplicities with which \([1^5] \) and \([3 2]\) appear in the representation of \( S_5 \) on the 0-weight space of \( V \).
Poincaré series

plethysms

[Littlewood, p. 204: "induced matrix of an invariant matrix"]

\[ \rho : GL_n \rightarrow GL_m \text{ (irreducible representation)} \]
\[ \sigma : GL_m \rightarrow GL_p \text{ (irreducible representation)} \]
\[ (\sigma \circ \rho) : GL_n \rightarrow GL_p \text{ (reducible representation)} \]

process to decompose into irreducibles, plethysm

[Gay, Gutkin] \( \mu \) : representation of \( S_5 \) corresponding to \([1^5] \) or \([3 \ 2] \).

Consider \( H = S_d \times S_d \times S_d \times S_d \times S_d \). Then, \( N_{S_{5d}}(H)/H \simeq S_5 \).

\( \mu \) representation of \( S_5 \), is representation of \( N_{S_{5d}}(H) \)

the multiplicity with which \( \mu = [1^5] \) or \([3 \ 2] \) appears in the representation of \( S_5 \) on \( V_0 \) is the multiplicity with which \( [(w+1) \ w] \) appears in the representation \( \check{\mu}^{S_{5d}} \) of \( S_{5d} \) induced from \( \mu \)

This is a plethysm [Macdonald, pp.135/6] denoted by \([1^5] \circ [d] \) (resp. \([3 \ 2] \circ [d] \)).
There are special features of this plethysm which greatly simplify the usual calculations. For example, we obtain the following results:

<table>
<thead>
<tr>
<th>$w$</th>
<th>multiplicity of $[1^5]$</th>
<th>multiplicity of $[3\ 2]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>17</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>22</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>27</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>32</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>507</td>
<td>425</td>
<td>2176</td>
</tr>
<tr>
<td>10842</td>
<td>195843</td>
<td>980298</td>
</tr>
</tbody>
</table>

From the standpoint of solving equations, the representation $[1^5]$ is not interesting; the corresponding resolvent is $a(x - by)^6$.

**Theorem.** Let $w \equiv 2(\text{mod } 5)$ and $d = \frac{2w+1}{5}$. Let $\rho : GL_5 \rightarrow GL(V)$ be the irreducible representation having highest weight $(w+1)\chi_1 + w\chi_2$. The vector space consisting of all $\overline{F} \in \mathbb{C}[M_{2,5}]$ such that

1. $\overline{F}$ is homogenous of degree $d$ in each column,
2. $\overline{F}$ is left $U$-invariant,
3. $\overline{F}$ has left $T$-weight 1,
4. $(12345)\overline{F} = \overline{F}$ and $(2354)\overline{F} = -\overline{F}$

is isomorphic to the vector space consisting of vectors $v$ in the $0$-weight space of $V$ which satisfy $(12345)v = v$ and $(2354)v = -v$.

The dimension of this vector space is the sum of the multiplicities with which $[1^5]$ and $[3\ 2]$ appear in the representation of $S_5$ on the $0$-weight space of $V$. The dimension can be found by calculating the plethysms $[1^5] \circ [d]$ and $[3\ 2] \circ [d]$. 

30
Using the Theorems and plethysm considerations, can show there are infinitely many non-equivalent covariants of Perrin-McClintock type (Setting I).

It also seems likely that there are infinitely many non-equivalent covariants of Perrin-McClintock type for which $\Psi/\Phi$ is fixed by $F_{20}$ and not by $S_5$ so we get resolvents for deciding solvability.
References


Malfatti, G. De AEquationibus Quadrato-cubicis Disquisitio Analytica, Siena transactions 1771

McClintock, E. Analysis of quintic equations, American J Math, Vol. 8, No. 1, Sep., 1885, 45 - 84

Perrin, Sur les cas de resolubilite par radicaux de l’equation du cinquieme degree Bulletin de la SMF Tome 11 1883 pp. 61 - 65