The Perrin-McClintock Resolvent, Solvable Quintics and Plethysms

Frank Grosshans

In his seminal paper of 1771, Lagrange found that certain polynomials of degree 6 called resolvents could be used to determine whether a quintic polynomial was solvable in radicals. Among the various resolvents later discovered, the Perrin-McClintock resolvent has some particularly noteworthy properties. We shall discuss these properties and their application to solvable quintics. The properties suggest that the Perrin-McClintock resolvent may be unique. We discuss this question and relate it to the representation theory of the general linear group, especially zero weight spaces and plethysms of a special form.

Properties of the Perrin-McClintock Resolvent

Property 1: a polynomial function

$d = 1, 2, \ldots$

$$\begin{aligned} f(x,y) &= \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i, \qquad a_i \in \mathbb{C} \\ &= a_0 x^d + \binom{d}{1} a_1 x^{d-1} y + \binom{d}{2} a_2 x^{d-2} y^2 + \ldots + \binom{d}{d} a_d y^d \\ &\leftrightarrow (a_0, a_1, \ldots, a_d) \end{aligned}$$

V_d is vector space over $\mathbb{C}$ spanned by all such $f(x,y)$

$\mathbb{A}^2 = \mathbb{C}^2 = \left\{ \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \right\}$

The Perrin-McClintock resolvent

a polynomial, $K : V_5 \times \mathbb{A}^2 \to \mathbb{C}$

$$K(a_0, a_1, a_2, a_3, a_4, a_5; x, y) = \sum_{j=0}^{6} \kappa_j(a_0, a_1, a_2, a_3, a_4, a_5) x^{6-j} y^j$$

$R_f(x) = K(f, \begin{pmatrix} x \\ 1 \end{pmatrix})$

Example 1: $f(x) = x^5 + 10a_2x^3 + 5a_4x + a_5$ with $a_4 = 4a_2^2$

$$K(f,v) = (3a_2^6 + a_2a_5^2)x^6 - 125a_2^4a_5x^5y + (4080a_2^7 - 15a_2^2a_5^2)x^4y^2$$

$$+1000a_2^5a_5x^3y^3 + (960a_2^8 + 70a_2^3a_5^2)x^2y^4 + (128a_2^6a_5 + a_2a_5^3)xy^5$$

Example 2: $f(x) = x^5 + 5x^4 + 9x^3 + 5x^2 - 4x - 5$

$$K(f,v) = \frac{1}{80000}(-498x^6 - 5900x^5y - 22662x^4y^2 - 41320x^3y^3 - 36254x^2y^4$$

$$-6860xy^5 + 8150y^6)$$

Example 3: $f(x) = x^5 - 8x^4 + 5x^3 - 6x^2 + 8x - 4$

$$K(f,v) = -5681513x^6 + 22679884x^5y - 42714844x^4y^2 + 6325088x^3y^3$$

$$+16299792x^2y^4 - 18575936xy^5 + 5294016y^6$$

Properties of the Perrin-McClintock Resolvent

Property 2: a covariant

$SL(2,\mathbb{C}), g = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$

action on V_d

$g{\cdot}x = dx - by$

$g{\cdot}y = -cx + ay$

$$f = \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i \to g{\cdot}f = \sum_{i=0}^{d} \binom{d}{i} a_i (dx - by)^{d-i}(-cx + ay)^i$$

action on $\mathbb{A}^2$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} a\lambda + b\mu \\ c\lambda + d\mu \end{pmatrix}$

The covariant property

(1) $K(g{\cdot}f, g{\cdot}v) = K(f, v)$ for all $g \in SL(2,\mathbb{C})$, $f \in V_d$, $v \in \mathbb{A}^2$

(2) coefficients of x and y terms form irreducible representation of $SL(2,\mathbb{C})$

(3) *source* of covariant is $K(f, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$

completely determines K

Properties of the Perrin-McClintock Resolvent

The Hessian cubic covariant

$g \in SL(2,\mathbb{C}), g = \begin{pmatrix} 5 & 2 \\ 17 & 7 \end{pmatrix}$

action on V_3

$g{\cdot}x = 7x - 2y$

$g{\cdot}y = -17x + 5y$

$$\begin{aligned} f &= a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3 \rightarrow \\ g{\cdot}f &= a_0(7x-2y)^3 + 3a_1(7x-2y)^2(-17x+5y) + 3a_2(7x-2y)(-17x+5y)^2 + a_3(-17x+5y)^3 \end{aligned}$$

action on $\mathbb{A}^2$

$\begin{pmatrix} 5 & 2 \\ 17 & 7 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 5\lambda + 2\mu \\ 17\lambda + 7\mu \end{pmatrix}$

$H(a_0, a_1, a_2, a_3; x, y) = \frac{1}{36} Det \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

$= (a_0a_2 - a_1^2)x^2 + (a_0a_3 - a_1a_2)xy + (a_1a_3 - a_2^2)y^2$

$$g = \begin{pmatrix} 5 & 2 \\ 17 & 7 \end{pmatrix}$$

$$f = 4x^3 + 3 \times 5x^2y + 3 \times (-6)xy^2 + (-1)y^3$$

$$g{\cdot}f = -42624x^3 + 37128x^2y - 10779xy^2 + 1043y^3$$

$$v = \begin{pmatrix} 8 \\ 3 \end{pmatrix};\ g{\cdot}v = \begin{pmatrix} 46 \\ 157 \end{pmatrix}$$

The covariant property

(1) $H(g{\cdot}f, g{\cdot}v) = H(f, v)$ for all $g \in G,\ f \in V_d,\ v \in \mathbb{A}^2$

$H(f, v) = H(4, 5, -6, -1; 8, 3) = -2881$

$H(g{\cdot}f, g{\cdot}v) = H(-42624, 12376, -3593, 1043; 46, 157)$

$= -2881$

(2) coefficients of x and y terms form irreducible representation of $SL(2, \mathbb{C})$

$(a_0a_2 - a_1^2), (a_0a_3 - a_1a_2), (a_1a_3 - a_2^2)$

(3) source of covariant is $H(f, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = a_0a_2 - a_1^2$

completely determines H

algebraic meaning: $H(f, v) = 0$ for all $v \in \mathbb{A}^2$ if and only if there is a linear form, say $g = ax + by$, such that $f = g^3$.

For example, $f(x, y) = 64x^3 - 144x^2y + 108xy^2 - 27y^3$.

$H(f, v) \equiv 0,\ f(x, y) = (4x - 3y)^3$

[Abdesselam and Chipalkatti]

Properties of the Perrin-McClintock Resolvent

Property 3: solvable quintics

Theorem. Let $f(x) = a_0x^5 + 5a_1x^4 + 10a_2x^3 + 10a_3x^2 + 5a_4x + a_5$ be an irreducible quintic polynomial in $\mathbb{Q}[x]$. Then $f(x)$ is solvable in radicals if and only if $R_f(x)$ has a rational root or is of degree 5.

Example 2: $f(x) = x^5 + 5x^4 + 9x^3 + 5x^2 - 4x - 5$

$$K(f,v) = \tfrac{1}{80000}(-498x^6 - 5900x^5y - 22662x^4y^2 - 41320x^3y^3 - 36254x^2y^4 \\ -6860xy^5 + 8150y^6)$$

$$R_f(x) = \tfrac{1}{80000}(-498x^6 - 5900x^5 - 22662x^4 - 41320x^3 - 36254x^2 \\ -6860x + 8150)$$

has root 1/3. Hence, $f(x)$ is solvable in radicals.

Example 3: $f(x) = x^5 - 8x^4 + 5x^3 - 6x^2 + 8x - 4$

$$K(f,v) = -5681513x^6 + 22679884x^5y - 42714844x^4y^2 + 6325088x^3y^3 + \\ 16299792x^2y^4 - 18575936xy^5 + 5294016y^6$$

$$R_f(x) = -5681513x^6 + 22679884x^5 - 42714844x^4 + 6325088x^3 + \\ 16299792x^2 - 18575936x + 5294016$$

does not have a rational root. Hence, $f(x)$ is *not* solvable in radicals.

Get elegant way to find solutions in radicals

Cayley to McClintock (McClintock, p.163): "McClintock completes in a very elegant manner the determination of the roots of the quintic equation . . . ."

Properties of the Perrin-McClintock Resolvent

Property 4: global information

Problem: use resolvents to obtain global information about solvable quintics

Example 1 (Perrin):

$$\begin{aligned} f(x) &= a_0x^5 + 10a_2x^3 + 5a_4x + a_5 \\ \text{with } a_4 &= 4a_2^2 \end{aligned}$$

$$R_f(x) = (3a_2^6 + a_2a_5^2)x^6 - 125a_2^4a_5x^5 + (4080a_2^7 - 15a_2^2a_5^2)x^4$$

$$+1000a_2^5a_5x^3 + (960a_2^8 + 70a_2^3a_5^2)x^2 + (128a_2^6a_5 + a_2a_5^3)x$$

has root 0. Hence, $f(x)$ is solvable in radicals.

Example 2: the McClintock parametrization

Have mapping φ, a rational function,

$$\begin{aligned} \varphi &: \mathbb{A}^4_{\mathbb{Q}} \to \mathbb{A}^4_{\mathbb{Q}} \\ (p, r, w, t) &\to (\gamma, \delta, \varepsilon, \zeta) \\ (\gamma, \delta, \varepsilon, \zeta) \text{ identified with } f(x) &= x^5 + 10\gamma x^3 + 10\delta x^2 + 5\varepsilon x + \zeta \end{aligned}$$

The polynomial $f(x)$ is solvable (its resolvent $R_f(x)$ has t as a root).

inverse map exists, rational function
need $R_f(x)$ to have rational root t

difficulty: if quintic factors, t may be complex or irrational real
so don't quite parametrize all solvable quintics

Example 3: Brioschi quintics [Elia]

$$f(x) = x^5 - 10zx^3 + 45z^2x - z^2$$

$$\begin{aligned} R_f(x) &= (-z^5 + 128z^6)x^6 + 400z^6x^5 + (-15z^6 - 46080z^7)x^4 \\ &\quad +40000z^7x^3 + (-95z^7 - 51840z^8)x^2 \\ &\quad +(z^7 + 1872z^8)x - 25z^8 \end{aligned}$$

If z is a non-zero integer, then $f(x)$ is solvable in radicals.

Example 4: subject to certain explicitly defined polynomials not vanishing, if f_0 is an irreducible quintic such that R_{f_0} has a root $t_0 \in \mathbb{R}$, then every Euclidean open neighborhood of f_0 contains a solvable quintic.

Dickson's Factorization

Action of S_5

S_5: symmetric group on 5 letters

Action of S_5 on polynomials

$$f(x_1, x_2, x_3, x_4, x_5) \in \mathbb{Z}[x_1, x_2, x_3, x_4, x_5]$$

$$\sigma \in S_5$$

$$\sigma \cdot f = f(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3}, x_{\sigma 4}, x_{\sigma 5})$$

Example

$$f = x_1x_2 - x_1x_3 + x_2x_3 - x_1x_4 - x_2x_4 + x_3x_4 + x_1x_5 - x_2x_5 - x_3x_5 + x_4x_5$$

$$\sigma = (132)$$

$$\sigma \cdot f = x_3x_1 - x_3x_2 + x_1x_2 - x_3x_4 - x_1x_4 + x_2x_4 + x_3x_5 - x_1x_5 - x_2x_5 + x_4x_5$$

Dickson's Factorization

The group F_{20}

S_5: symmetric group on 5 letters

F_{20}: subgroup of S_5 generated by (12345) and (2354)

$S_5 = \bigcup_{i=1}^{6} \tau_i F_{20}$

$$\tau_1 = (1),\ \tau_2 = (12),\ \tau_3 = (13),\ \tau_4 = (23),\ \tau_5 = (123),\ \tau_6 = (132)$$

Theorem. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible quintic. Then $f(x)$ is solvable in radicals if and only if its Galois group is conjugate to a subgroup of F_{20}.

Problem: extend resolvent program to polynomials of higher degree. (What replaces F_{20}?)

Dickson's Factorization

Malfatti's resolvent

$$\begin{aligned}\Phi(x_1,x_2,x_3,x_4,x_5) &= x_1x_2 - x_1x_3 + x_2x_3 - x_1x_4 - x_2x_4 + x_3x_4 + x_1x_5 - x_2x_5 - x_3x_5 + x_4x_5 \\ &= (x_1-x_5)(x_2-x_5) + (x_2-x_5)(x_3-x_5) + (x_3-x_5)(x_4-x_5) \\ &\quad -(x_2-x_5)(x_4-x_5) - (x_4-x_5)(x_1-x_5) - (x_1-x_5)(x_3-x_5)\end{aligned}$$

Properties:

(1) homogeneous of degree 2 in x_1, x_2, x_3, x_4, x_5

(2) for any $\beta \in \mathbb{C}$, $\Phi(x_1+\beta, x_2+\beta, x_3+\beta, x_4+\beta, x_5+\beta) = \Phi(x_1, x_2, x_3, x_4, x_5)$

(3) highest power to which any x_i appears is 1

(4) $(12345)\Phi = \Phi$
$(2354)\Phi = -\Phi$

Note: Malfatti resolvent, put $\Phi^2 = \Theta$

$$R(x) = (x-\Theta)(x-\tau_2\Theta)(x-\tau_3\Theta)(x-\tau_4\Theta)(x-\tau_5\Theta)(x-\tau_6\Theta)$$

polynomial in $a_i's$
rational root if and only if $f(x)$ solvable in radicals
for resolvents of this form, lowest possible degree in Θ
rediscovered by Jacobi (1835), Cayley (1861), Dummit (1991)

Dickson's Factorization

Roots of the resolvent

$$S_5 = \bigcup_{i=1}^{6} \tau_i F_{20}$$

$$\tau_1 = (1),\ \tau_2 = (12),\ \tau_3 = (13),\ \tau_4 = (23),\ \tau_5 = (123),\ \tau_6 = (132)$$

The Malfatti resolvent

$$\begin{aligned}
\Phi(x_1, x_2, x_3, x_4, x_5) &= x_1x_2 - x_1x_3 + x_2x_3 - x_1x_4 - x_2x_4 + x_3x_4 + x_1x_5 - x_2x_5 - x_3x_5 + x_4x_5 \\
&= (x_1 - x_5)(x_2 - x_5) + (x_2 - x_5)(x_3 - x_5) + (x_3 - x_5)(x_4 - x_5) \\
&\quad -(x_2 - x_5)(x_4 - x_5) - (x_4 - x_5)(x_1 - x_5) - (x_1 - x_5)(x_3 - x_5)
\end{aligned}$$

$$\Psi(x_1, x_2, x_3, x_4, x_5) = (x_1x_2x_3x_4x_5)\Phi(1/x_1, 1/x_2, 1/x_3, 1/x_4, 1/x_5)$$

where does Ψ come from?

need highest power to which a root appears in Φ is ≤ 1

homogeneous of degree 3 in x_1, x_2, x_3, x_4, x_5

Perrin-McClintock resolvent: for $i = 1, 2, 3, 4, 5, 6$, put $\Phi_i = \tau_i\Phi$, $\Psi_i = \tau_i\Psi$

$$\begin{aligned}
K(f; v) &= a_0^6 \prod_{i=1}^{6}((\tau_i\Phi)x - (\tau_i\Psi)y) \\
&= a_0^6(\prod_{i=1}^{6}(\tau_i\Phi)) \prod_{i=1}^{6}(x - ((\tau_i\Psi)/(\tau_i\Phi))y)
\end{aligned}$$

Constructing resolvents

Setting I: covariants

Find covariants of the form

$$K(f;v) = a_0^m \prod_{i=1}^{6}((\tau_i \Phi)x - (\tau_i \Psi)y)$$

with

(1) $\Phi(x_1, x_2, x_3, x_4, x_5)$ homogeneous of degree $w \equiv 2 (\bmod 5)$ in x_1, x_2, x_3, x_4, x_5

(2) highest power to which a root appears in Φ is $\leq d = \frac{2w+1}{5}$

(3) for any $\beta \in \mathbb{C}$, $\Phi(x_1+\beta, x_2+\beta, x_3+\beta, x_4+\beta, x_5+\beta) = \Phi(x_1, x_2, x_3, x_4, x_5)$

(4) $(12345)\Phi = \Phi$
$(2354)\Phi = -\Phi$

$$\Psi(x_1, x_2, x_3, x_4, x_5) = (x_1 x_2 x_3 x_4 x_5)^d \Phi(-1/x_1, -1/x_2, -1/x_3, -1/x_4, -1/x_5)$$

Recall: source determines covariant.

source is $a_0^m \prod_{i=1}^{6}((\tau_i \Phi)$

Constructing resolvents

Setting II: polynomials in roots

For $w \equiv 2 (\mathrm{mod}\, 5)$, put $d = \frac{2w+1}{5}$. Find $\Phi(x_1, x_2, x_3, x_4, x_5)$

(1) homogeneous of degree w in x_1, x_2, x_3, x_4, x_5

(2) the highest power to which any x_i appears in Φ is $\leq d$

(3) for any $\beta \in \mathbb{C}$, $\Phi(x_1+\beta, x_2+\beta, x_3+\beta, x_4+\beta, x_5+\beta) = \Phi(x_1, x_2, x_3, x_4, x_5)$

(4) $(12345)\Phi = \Phi$
$(2354)\Phi = -\Phi$

For covariant, need $w \equiv 2 (\mathrm{mod}\, 5)$

Malfatti is only such polynomial of degree 2

Problems.

1. no $SL_2(\mathbb{C})$ action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_i = (dx_i - b)/(-cx_i + a)$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot x_i = -1/x_i$$

2. the highest power to which any x_i appears in Φ is $\leq d$

$f(x_1, \ldots, x_p) \in \mathbb{Z}[x_1, \ldots, x_p],\ f = \sum \lambda_{(a)} x_1^{a_1} \ldots x_p^{a_p}$

$\mathcal{R}(f) = \max\{a_i : 1 \leq i \leq p, \lambda_{(a)} \neq 0\}$

If $f(x_1, \ldots, x_p)$ is symmetric in $x_1, \ldots, x_p$, then $f(x_1, \ldots, x_p) = \sum \tau_{(b)} \sigma_1^{b_1} \ldots \sigma_p^{b_p}$
$\sigma_i = ith$ elementary symmetric function.

$\mathcal{D}(f) = \max\{b_1 + \ldots + b_p : \tau_{(b)} \neq 0\}$

Theorem. If $f(x_1, \ldots, x_p)$ is symmetric in $x_1, \ldots, x_p$, then $\mathcal{R}(f) = \mathcal{D}(f)$.

Constructing resolvents

Setting III: matrix variables

Translation:

For $d \equiv 1 (\bmod 2)$, find matrix polynomials $\widetilde{F}\left(\begin{pmatrix} \widetilde{x}_1 & \widetilde{x}_2 & \widetilde{x}_3 & \widetilde{x}_4 & \widetilde{x}_5 \\ \widetilde{y}_1 & \widetilde{y}_2 & \widetilde{y}_3 & \widetilde{y}_4 & \widetilde{y}_5 \end{pmatrix}\right)$,

(1) $\widetilde{F}$ is homogenous of degree d in each column, i.e.,

$$\widetilde{F} = \sum c_{(e)} \widetilde{x}_1^{e_1} \widetilde{y}_1^{d-e_1} \ldots \widetilde{x}_5^{e_5} \widetilde{y}_5^{d-e_5}$$

(2) $\widetilde{F}$ is left U-invariant, i.e.,

$$\widetilde{F}\left(\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \widetilde{x}_1 & \widetilde{x}_2 & \widetilde{x}_3 & \widetilde{x}_4 & \widetilde{x}_5 \\ \widetilde{y}_1 & \widetilde{y}_2 & \widetilde{y}_3 & \widetilde{y}_4 & \widetilde{y}_5 \end{pmatrix}\right) = \widetilde{F}\begin{pmatrix} \widetilde{x}_1 & \widetilde{x}_2 & \widetilde{x}_3 & \widetilde{x}_4 & \widetilde{x}_5 \\ \widetilde{y}_1 & \widetilde{y}_2 & \widetilde{y}_3 & \widetilde{y}_4 & \widetilde{y}_5 \end{pmatrix}$$

or

$$\sum c_{(e)} (\widetilde{x}_1 + \beta \widetilde{y}_1)^{e_1} \widetilde{y}_1^{d-e_1} \ldots (\widetilde{x}_5 + \beta \widetilde{y}_5)^{e_5} \widetilde{y}_5^{d-e_5} = \sum c_{(e)} \widetilde{x}_1^{e_1} \widetilde{y}_1^{d-e_1} \ldots \widetilde{x}_5^{e_5} \widetilde{y}_5^{d-e_5}$$

(3) $\widetilde{F}$ has left T-weight 1, i.e., $5d - 2(e_1 + e_2 + e_4 + e_4 + e_5) = 1$

or

$$\widetilde{F}\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}\begin{pmatrix} \widetilde{x}_1 & \widetilde{x}_2 & \widetilde{x}_3 & \widetilde{x}_4 & \widetilde{x}_5 \\ \widetilde{y}_1 & \widetilde{y}_2 & \widetilde{y}_3 & \widetilde{y}_4 & \widetilde{y}_5 \end{pmatrix}\right) = \tfrac{1}{\lambda}\widetilde{F}\begin{pmatrix} \widetilde{x}_1 & \widetilde{x}_2 & \widetilde{x}_3 & \widetilde{x}_4 & \widetilde{x}_5 \\ \widetilde{y}_1 & \widetilde{y}_2 & \widetilde{y}_3 & \widetilde{y}_4 & \widetilde{y}_5 \end{pmatrix}$$

or

$$\sum c_{(e)}(\lambda\widetilde{x}_1 + \frac{1}{\lambda}\widetilde{y}_1)^{e_1}\widetilde{y}_1^{d-e_1} \ldots (\lambda\widetilde{x}_5 + \frac{1}{\lambda}\widetilde{y}_5)^{e_5}\widetilde{y}_5^{d-e_5} = \frac{1}{\lambda}\sum c_{(e)}\widetilde{x}_1^{e_1}\widetilde{y}_1^{d-e_1} \ldots \widetilde{x}_5^{e_5}\widetilde{y}_5^{d-e_5}$$

(4) S_5 acts on vector variables by permuting columns

$(12345)\widetilde{F} = \widetilde{F}$
$(2354)\widetilde{F} = -\widetilde{F}$

$$\widetilde{F}\left(\begin{pmatrix} \widetilde{x}_2 & \widetilde{x}_3 & \widetilde{x}_4 & \widetilde{x}_5 & \widetilde{x}_1 \\ \widetilde{y}_2 & \widetilde{y}_3 & \widetilde{y}_4 & \widetilde{y}_5 & \widetilde{y}_1 \end{pmatrix}\right) = \widetilde{F}\left(\begin{pmatrix} \widetilde{x}_1 & \widetilde{x}_2 & \widetilde{x}_3 & \widetilde{x}_4 & \widetilde{x}_5 \\ \widetilde{y}_1 & \widetilde{y}_2 & \widetilde{y}_3 & \widetilde{y}_4 & \widetilde{y}_5 \end{pmatrix}\right)$$

$$\widetilde{F}\left(\begin{pmatrix} \widetilde{x}_1 & \widetilde{x}_3 & \widetilde{x}_5 & \widetilde{x}_2 & \widetilde{x}_4 \\ \widetilde{y}_1 & \widetilde{y}_3 & \widetilde{y}_5 & \widetilde{y}_2 & \widetilde{y}_4 \end{pmatrix}\right) = -\widetilde{F}\left(\begin{pmatrix} \widetilde{x}_1 & \widetilde{x}_2 & \widetilde{x}_3 & \widetilde{x}_4 & \widetilde{x}_5 \\ \widetilde{y}_1 & \widetilde{y}_2 & \widetilde{y}_3 & \widetilde{y}_4 & \widetilde{y}_5 \end{pmatrix}\right)$$

Constructing resolvents

equivalences

Setting I: find covariants of the form $K(f;v) = a_0^m \prod_{i=1}^{6} ((\tau_i \Phi)x - (\tau_i \Psi)y)$

Setting II: find $\Phi(x_1, x_2, x_3, x_4, x_5)$

Setting III: find matrix polynomials $\widetilde{F}$

Definition. For $i = 1, 2$, let $K_i : V_5 \times \mathbb{A}^2 \to \mathbb{C}$ be covariants as in Setting I. Say $K_1 \sim K_2$ if and only if there are $\widetilde{\mu}$, $\widetilde{\rho} \in \mathbb{C}[V_5]^{SL_2(\mathbb{C})}$ with $\widetilde{\mu} K_1(x, y) = \widetilde{\rho} K_2(x, y)$.

Let Φ and Φ' be as in Setting II. Say $\Phi \sim \Phi'$ if and only if $\frac{\Psi}{\Phi} = \frac{\Psi'}{\Phi'}$.

Setting I and Setting II:

$K = a_0^m \prod_{i=1}^{6} ((\tau_i \Phi)x - (\tau_i \Psi)y)$, $K' = a_0^{m'} \prod_{i=1}^{6} ((\tau_i \Phi')x - (\tau_i \Psi')y)$

$K \sim K'$ if and only if $\Phi \sim \Phi'$

Setting II and Setting III: there is vector space isomorphism between

Φ homogeneous of degree w

$\widetilde{F}$ homogeneous of degree $2w + 1$

also, have algebra homomorphism

have mapping $\Omega : \mathbb{C}\begin{bmatrix} \widetilde{x}_1 & \widetilde{x}_2 & \widetilde{x}_3 & \widetilde{x}_4 & \widetilde{x}_5 \\ \widetilde{y}_1 & \widetilde{y}_2 & \widetilde{y}_3 & \widetilde{y}_4 & \widetilde{y}_5 \end{bmatrix} \to \mathbb{C}[x_1, x_2, x_4, x_4, x_5]$

$\widetilde{x}_i \to x_i, \widetilde{y}_i \to 1$

Constructing resolvents

finitely generated modules

$R_w \subset \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$

For $w \equiv 0 (\mathrm{mod}\, 5)$, $w \geq 0$, R_w is vector space spanned by all linear combinations Φ of products $(x_{i_1} - x_{j_1}) \dots (x_{i_w} - x_{j_w})$ such that

(1) each x_i appears $\frac{2w}{5}$ times in every product

(2) $(12345)\Phi = \Phi$, $(2354)\Phi = \Phi$

$R = \bigoplus R_w$

$M_w \subset \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$

For $w \equiv 2 (\mathrm{mod}\, 5)$, $w \geq 0$, M_w is vector space spanned by $\Phi(x_1, x_2, x_3, x_4, x_5)$

(1) homogeneous of degree w in x_1, x_2, x_3, x_4, x_5

(2) the highest power to which any x_i appears in Φ is $\leq \frac{2w+1}{5}$

(3) for any $\beta \in \mathbb{C}$, $\Phi(x_1{+}\beta, x_2{+}\beta, x_3{+}\beta, x_4{+}\beta, x_5{+}\beta) = \Phi(x_1, x_2, x_3, x_4, x_5)$

(4) $(12345)\Phi = \Phi$, $(2354)\Phi = -\Phi$

$M = \bigoplus M_w$

Theorem. (a) R is finitely generated $\mathbb{C}$-algebra.

(b) $\Delta = \Phi_2\Psi_7 - \Phi_7\Psi_2 \neq 0$ and is in R.

(c) $\Phi \in M, \Phi = \frac{r_1}{\Delta}\Phi_2 + \frac{r_2}{\Delta}\Phi_7$

(d) M is finitely generated R-module

(e) $\dim_{Q(R)} M \otimes_R Q(R) = 2$

Poincaré series

Hilbert - Serre theorem

Recall $R = \bigoplus R_w$, is finitely generated $\mathbb{C}$-algebra

$M = \bigoplus M_w$, M_w polynomials as in Setting II
is finitely generated R-module

Poincaré series: $P(M,t) = \sum_{w \equiv 2 (\bmod 5)}^{\infty} \dim M_w$

Theorem (Hilbert, Serre, applied here). Let γ be the number of generators of R. Then

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{\gamma}(1 - t^{d_i})}$$

for suitable positive integers d_i and $f(t) \in \mathbb{Z}[t]$.

Problem: determine $P(M,t)$.

Determine $\dim M_w$.

Poincaré series

$GL_m - GL_n$ duality

to understand:

(2) $\widetilde{F}$ is left U-invariant

(3) $\widetilde{F}$ has left T-weight 1

$T_r \subset GL_r$; subgroup consisting of diagonal matrices

$U_r \subset GL_r$; subgroup consisting of upper triangular matrices, 1's on diagonal

A highest weight of an irreducible polynomial representation of GL_r with respect to the Borel subgroup T_rU_r is a character of the form $\chi = e_1\chi_1 + \cdots + e_r\chi_r$ where $e_1 \geq \ldots \geq e_r \geq 0$. If e_ℓ is the last non-zero e_i, we say that the highest weight χ has *depth* ℓ.

Theorem ($GL_m - GL_n$ duality) [Howe, Section 2.1.2]. Let U and V be finite-dimensional vector spaces over $\mathbb{C}$. The symmetric algebra $\mathcal{S}(U \otimes V)$ is multiplicity-free as a $GL(U) \times GL(V)$ module. Precisely, we have a decomposition

$$\mathcal{S}(U \otimes V) = \sum_D \rho_U^D \otimes \rho_V^D$$

of $GL(U) \times GL(V)$-modules. Here D varies over all highest weights of depth at most $min\{dimU, dimV\}$.

Translation

$M_{2,5}$: the algebra consisting of all 2×5 matrices with entries in $\mathbb{C}$.

GL_2 acts on $M_{2,5}$ by left multiplication: $g{\cdot}m = gm$ for all $g \in GL_2$ and $m \in M_{2,5}$.

GL_5 acts on $M_{2,5}$ by right multiplication: $g{\cdot}m = mg^{-1}$ for all $g \in GL_5$ and $m \in M_{2,5}$.

These actions commute and give an action of $G = GL_2 \times GL_5$ on $M_{2,5}$ and $\mathbb{C}[M_{2,5}]$.

$M_{2,5} \leftrightarrow \mathbb{A}^2 \otimes (\mathbb{A}^5)^*$

$\mathbb{C}[M_{2,5}] \leftrightarrow S((\mathbb{A}^2)^* \otimes \mathbb{A}^5)$

Suppose that $d \equiv 1 (\bmod 2)$, $5d = 2w + 1$ and that

(2) $\widetilde{F}$ is left U-invariant

(3) $\widetilde{F}$ has left T-weight 1

then: the terms $\widetilde{F} = \widetilde{v} \otimes V_D$ appear when

$\widetilde{v}$: highest weight vector of irreducible representation GL_2, highest weight $(w+1)\chi_1 + w\chi_2$

ρ_V^D is irreducible representation of GL_5, highest weight $(w+1)\chi_1 + w\chi_2$

Note. can explicitly construct the invariants $\widetilde{F}$ in terms of determinants using Young diagrams and straightening [Pommerening].

Poincaré series

Zero weight space

to understand:

(1) $\widetilde{F}$ is homogenous of degree d in each column

recall: $w \equiv 2 (\bmod 5)$, $5d = 2w + 1$

$$T_5 = \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix} \right\}, U_5 : \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 1 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 1 & a_{34} & a_{35} \\ 0 & 0 & 0 & 1 & a_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

$\rho : GL_5 \to GL(V)$

$V_0 = \{v \in V : \rho(t)v = (a_1a_2a_3a_4a_5)^e v\} = 0$ weight space of V

translation: $\widetilde{F} \in \mathbb{C}[M_{2,5}]$, $t \in T_5$, $m = (v_1, \ldots, v_5) \in M_{2,5}$

$$\begin{aligned} (t \cdot \widetilde{F})(v_1, \ldots, v_5) &= \widetilde{F}((v_1, \ldots, v_5)t) \\ &= \widetilde{F}(a_1v_1, \ldots, a_5v_5) \\ &= (a_1a_2a_3a_4a_5)^d \ \widetilde{F}(v_1, \ldots, v_5) \end{aligned}$$

Proposition. Let $w \equiv 2 \pmod 5$ and $d = \frac{2w+1}{5}$. Let $\rho : GL_5 \to GL(V)$ be the irreducible representation having highest weight $(w+1)\chi_1 + w\chi_2$. The vector space consisting of all $\widetilde{F} \in \mathbb{C}[M_{2,5}]$ such that

(1) $\widetilde{F}$ is homogenous of degree d in each column,

(2) $\widetilde{F}$ is left U-invariant,

(3) $\widetilde{F}$ has left T-weight 1

is isomorphic to the 0-weight space of V.

Poincaré series

S_5 action on zero weight space

to understand:

(4) $(12345)\widetilde{F} = \widetilde{F}$, $(2354)\widetilde{F} = -\widetilde{F}$

S_5 acts on $0-$weight space, V_0

$$V_0 = \bigoplus m_\chi V_\chi$$

V_χ runs over all irreducible representations of S_5

m_χ is multiplicity with which V_χ appears in V_0

S_5 has 7 irreducible representations

$[5]$, $[41]$, $[32]$, $[31^2]$, $[2^21]$, $[21^3]$, $[1^5]$

$\widetilde{\rho} : F_{20} \to \{\pm 1\}$

$\widetilde{\rho}(12345) = 1$
$\widetilde{\rho}(2354) = -1.$

$\widetilde{\rho}$ appears with multiplicity 1 in both $[3\ 2]$ and $[1^5]$. It does not appear in any of the other 5 irreducible representations.

Proposition. Let $w \equiv 2 (\bmod 5)$ and $d = \frac{2w+1}{5}$. Let $\rho : GL_5 \to GL(V)$ be the irreducible representation having highest weight $(w+1)\chi_1 + w\chi_2$. The vector space consisting of all $\widetilde{F} \in \mathbb{C}[M_{2,5}]$ such that

(1) $\widetilde{F}$ is homogenous of degree d in each column,

(2) $\widetilde{F}$ is left U-invariant,

(3) $\widetilde{F}$ has left T-weight 1,

(4) $(12345)\widetilde{F} = \widetilde{F}$ and $(2354)\widetilde{F} = -\widetilde{F}$

is isomorphic to the vector space consisting of vectors v in the 0-weight space of V which satisfy $(12345)v = v$ and $(2354)v = -v$.

The dimension of this vector space is the sum of the multiplicities with which $[1^5]$ and $[3\ 2]$ appear in the representation of S_5 on the 0-weight space of V.

Poincaré series

plethysms

[Littlewood, p. 204: "induced matrix of an invariant matrix"]
$\rho : GL_n \to GL_m$ (irreducible representation)
$\sigma : GL_m \to GL_p$ (irreducible representation)
$(\sigma \circ \rho) : GL_n \to GL_p$ (reducible representation)
process to decompose into irreducibles, plethysm

[Gay, Gutkin] μ : representation of S_5 corresponding to $[1^5]$ or [3 2].

Consider $H = S_d \times S_d \times S_d \times S_d \times S_d$. Then, $N_{S_{5d}}(H)/H \simeq S_5$.

μ representation of S_5, is representation of $N_{S_{5d}}(H)$

the multiplicity with which $\mu = [1^5]$ or [3 2] appears in the representation of S_5 on V_0 is the multiplicity with which $[(w+1)\ w]$ appears in the representation $\widehat{\mu}^{S_{5d}}$ of S_{5d} induced from μ

This is a plethysm [Macdonald, pp.135/6] denoted by $[1^5] \circ [d]$ (resp. $[3\ 2] \circ [d]$).

There are special features of this plethysm which greatly simplify the usual calculations. For example, we obtain the following results:

w	multiplicity of $[1^5]$	multiplicity of [3 2]
2	0	1
7	0	1
12	0	2
17	0	4
22	1	6
27	1	8
32	1	11
507	425	2176
10842	195843	980298

From the standpoint of solving equations, the representation $[1^5]$ is not interesting; the corresponding resolvent is $a(x-by)^6$.

Theorem. Let $w \equiv 2 (\text{mod}\, 5)$ and $d = \frac{2w+1}{5}$. Let $\rho : GL_5 \to GL(V)$ be the irreducible representation having highest weight $(w+1)\chi_1 + w\chi_2$. The vector space consisting of all $\widetilde{F} \in \mathbb{C}[M_{2,5}]$ such that

(1) $\widetilde{F}$ is homogenous of degree d in each column,

(2) $\widetilde{F}$ is left U-invariant,

(3) $\widetilde{F}$ has left T-weight 1,

(4) $(12345)\widetilde{F} = \widetilde{F}$ and $(2354)\widetilde{F} = -\widetilde{F}$

is isomorphic to the vector space consisting of vectors v in the 0-weight space of V which satisfy $(12345)v = v$ and $(2354)v = -v$.

The dimension of this vector space is the sum of the multiplicities with which $[1^5]$ and [3 2] appear in the representation of S_5 on the 0-weight space of V. The dimension can be found by calculating the plethysms $[1^5] \circ [d]$ and $[3\ 2] \circ [d]$.

Using the Theorems and plethysm considerations, can show there are infinitely many non-equivalent covariants of Perrin-McClintock type (Setting I).

It also seems likely that there are infinitely many non-equivalent covariants of Perrin-McClintock type for which Ψ/Φ is fixed by F_{20} and not by S_5 so we get resolvents for deciding solvability.

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