On a recursive decoding algorithm for lattices

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Workshop on lattices, codes and modular forms
Aachen, 27.09.2011

Overview

1. Introduction
2. Iterative lattice decoding
3. Upper bounds on the number of lattice points in a small sphere
4. Examples

Lattice Decoding: The Closest Vector Problem (CVP)

- Given a lattice L in $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, the CVP consists in finding $\ell \in L$ such that

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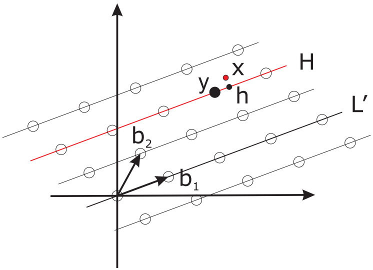
- The best known approximation factor for a deterministic polynomial time algorithm to solve the CVP approximately is $2^{n(\log\log n)^2/2\log n}$ (Schnorr 1985).

Babai's Nearest Plane Procedure (BNPP)



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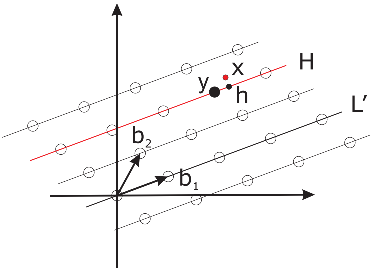

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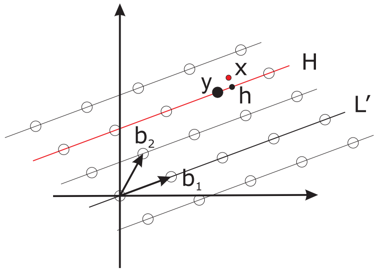

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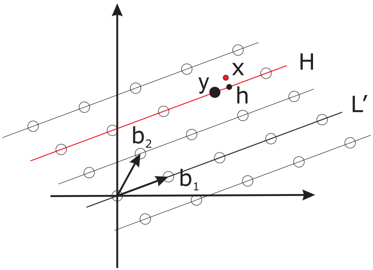

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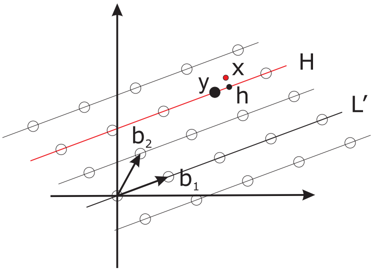

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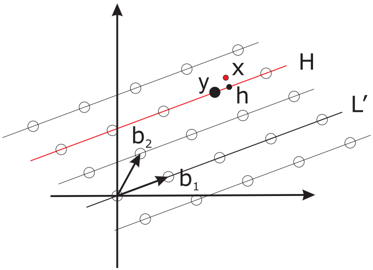

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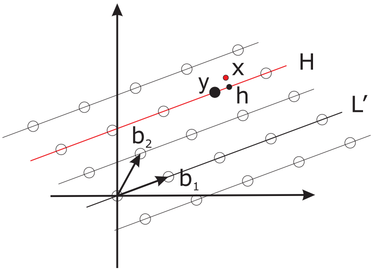

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Idea: Generalise BNPP, changing from lattices $\alpha\mathbb{Z}$ to higher dimensional lattices.

Iterative lattice decoding

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- *Sphere decoding* (Fincke, Pohst) can be used to compute $B_r(x_1) \cap W_1$.

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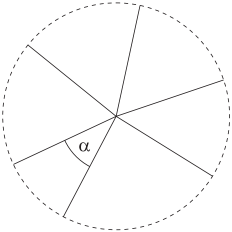
With $\delta_1, \ldots, \delta_{t-1}$ as above, Algorithm $\mathcal{A}$ achieves an approximation factor of

$$\max\{1 + \delta_{t-1}, r^{-1} \sum_{i=1}^{t} \mu(W_i)\}^{\frac{1}{2}}.$$

Question: Can we upper bound $|B_r(x_1) \cap W_1|$?

Bounds on $|B_r(x) \cap W|$ via spherical codes



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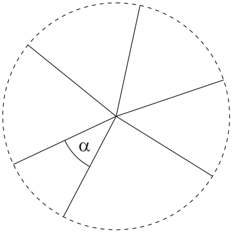

Definition

A *spherical code* in $\mathbb{R}^s$ is a set $\mathcal{C}$ of vectors of length 1. The minimum angle of $\mathcal{C}$ is $\alpha_{\min}(\mathcal{C}) := \min_{c \neq c' \in \mathcal{C}} \angle(c, c')$.

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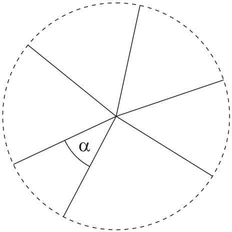

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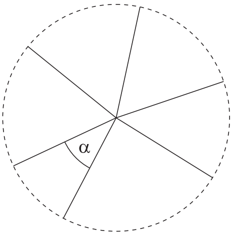

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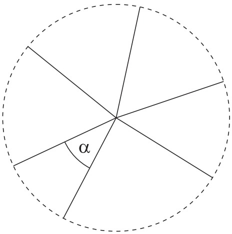

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Lattices and spherical codes

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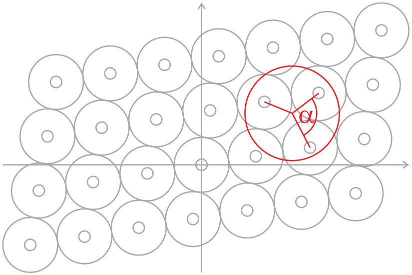

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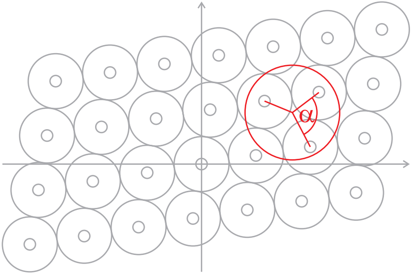

Theorem

Let L be a lattice in $\mathbb{R}^s$. If r is a real number with $0 < r \leq 2\rho_L$ then the set

$$\{|x - z|^{-1}(x - z) \mid z \in B_r(x) \cap L\}$$

is a spherical code with minimum angle $\alpha = \cos^{-1}(1 - \frac{\rho_L}{r})$, for every $x \in \mathbb{R}^s$.

Examples: Bounds obtained for A_n, E_n, Λ_{24}, $r = \gamma_L$

Type	n	θ	$A(n,\theta)$	Gaussian bound	for deep holes
A	2	$\frac{2}{3}\pi$	3	3	3
	3	$\frac{\pi}{2}$	6	7	6
	4	$\cos^{-1}(\frac{1}{6})$	10	12	10
	5	$\cos^{-1}(\frac{1}{3})$	≤ 24	26	20
	6	$\cos^{-1}(\frac{5}{12})$	≤ 54	47	35
	7	$\frac{\pi}{3}$	≤ 140	99	70
	8	-	-	188	126
	9	-	-	391	252
E	6	$\cos^{-1}(\frac{1}{4})$	27	37	27
	7	$\cos^{-1}(\frac{1}{3})$	56	84	56
	8	$\frac{\pi}{2}$	16	77	16
Leech	24	$\frac{\pi}{2}$	48	974	48

Example: Nebe's extremal even unimodular lattice Λ_{72}

- Λ_{72} is obtained from a polarisation $(\alpha(\Lambda_{24}), \beta(\Lambda_{24}))$ of the Leech lattice Λ_{24}, where $\alpha, \beta \in \mathrm{End}(\Lambda_{24})$ such that $\alpha^2 - \alpha + 2 = 0$, $\beta = 1 - \alpha$ and $(\alpha(x), y) = (x, \beta(y))$ for all $x, y \in \mathbb{R}^{24}$:

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- Algorithm $\mathcal{A}$: time increased by at most $|B_{\sqrt{2}}(x_1) \cap \Lambda_{24}| \leq 48$, approximation factor of $\sqrt{7}$, using sphere decoding with $r = \sqrt{2} = \sqrt{\mu(\Lambda_{24})}$.

Thank you very much for your attention!