The measurable chromatic number of Euclidean space

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Codes, lattices and modular forms
Aachen, September 26-29, 2011
The chromatic number of Euclidean space $\chi(\mathbb{R}^n)$ is the smallest number of colors needed to color every point of $\mathbb{R}^n$, such that two points at distance apart 1 receive different colors.

E. Nelson, 1950, introduced $\chi(\mathbb{R}^2)$.

Dimension 1:

$\chi(\mathbb{R}) = 2$

No other value is known!
\( \chi(\mathbb{R}^2) \leq 7 \)
\( \chi(\mathbb{R}^2) \leq 7 \)
\( \chi(\mathbb{R}^2) \leq 7 \)
\( \chi(\mathbb{R}^2) \geq 4 \)

**Figure:** The Moser’s Spindle
The two inequalities:

$$4 \leq \chi(\mathbb{R}^2) \leq 7$$

where proved by Nelson and Isbell, 1950. No improvements since then...
\( \chi(\mathbb{R}^n) \)

- Other dimensions: lower bounds based on

\[
\chi(\mathbb{R}^n) \geq \chi(G)
\]

for all finite graph \( G = (V, E) \) embedded in \( \mathbb{R}^n \) (\( G \rightarrow \mathbb{R}^n \)) i.e. such that \( V \subset \mathbb{R}^n \) and \( E = \{ (x, y) \in V^2 : \|x - y\| = 1 \} \).

- De Bruijn and Erdös (1951):

\[
\chi(\mathbb{R}^n) = \max_{G \text{ finite } G \rightarrow \mathbb{R}^n} \chi(G)
\]

\(\chi(\mathbb{R}^n)\) for large \(n\)

\[(1.2 + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n\]

- Lower bound: Frankl and Wilson (1981). Use graphs with vertices in \(\{0, 1\}^n\) and the “linear algebra method” to estimate \(\chi(G)\).
- FW 1.207\(n\) is improved to 1.239\(n\) by Raigorodskii (2000).
The measurable chromatic number $\chi_m(\mathbb{R}^n)$: the color classes are required to be measurable.

Obviously $\chi_m(\mathbb{R}^n) \geq \chi(\mathbb{R}^n)$.

Falconer (1981): $\chi_m(\mathbb{R}^n) \geq n + 3$. In particular

$$\chi_m(\mathbb{R}^2) \geq 5$$

The color classes are measurable 1-avoiding sets, i.e. contain no pair of points at distance apart 1.
\[ m_1(\mathbb{R}^n) = \sup \left\{ \delta(S) : S \subset \mathbb{R}^n, \ S \text{ measurable, avoids 1} \right\} \]

where \( \delta(S) \) is the density of \( S \):

\[ \delta(S) = \lim_{r \to +\infty} \sup \frac{\text{vol}(S \cap B_n(r))}{\text{vol}(B_n(r))}. \]

\[ \delta = 1/7 \]
Obviously

\[ \chi_m(\mathbb{R}^n) \geq \frac{1}{m_1(\mathbb{R}^n)} \]

Problem: to upper bound \( m_1(\mathbb{R}^n) \).

Larman and Rogers (1972):

\[ m_1(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|} \quad \text{for all } G \hookrightarrow \mathbb{R}^n \]

where \( \alpha(G) \) is the independence number of the graph \( G \) i.e. the max number of vertices pairwise not connected by an edge.
Finite graphs

- An independence set of a graph $G = (V, E)$ is a set of vertices pairwise not connected by an edge.
- The independence number $\alpha(G)$ of the graph is the number of elements of a maximal independent set.

- A 1-avoiding set in $\mathbb{R}^n$ is an independent set of the unit distance graph $V = \mathbb{R}^n$, $E = \{(x, y) : \|x - y\| = 1\}$. 
1-avoiding sets versus packings

$S$ avoids $d = 1$

$\delta(S) = \lim \frac{\text{vol}(S \cap B_n(r))}{\text{vol}(B_n(r))}$

$m_1(\mathbb{R}^n) = \sup_S \delta(S)$?

$S$ avoids $d \in ]0, 2[$

$\delta(S) = \lim \frac{|S \cap B_n(r)|}{\text{vol}(B_n(r))}$

$\delta_n = \sup_S \delta(S)$?

$S$ avoids $d = 1$

$\delta(S) = \frac{|S|}{|V|}$

$\frac{\alpha(G)}{|V|} = \sup_S \delta(S)$?
The linear programming method

- A general method to obtain upper bounds for densities of distances avoiding sets.
- **For packing problems:** initiated by Delsarte, Goethels, Seidel on $S^{n-1}$ (1977); Kabatianskii and Levenshtein on compact 2-point homogeneous spaces (1978); Cohn and Elkies on $\mathbb{R}^n$ (2003).
- **For finite graphs:** Lovász theta number $\vartheta(G)$ (1979).
- **For sets avoiding one distance:** B. G. Nebe, F. Oliveira, F. Vallentin for $m(S^{n-1}, \theta)$ (2009). F. Oliveira and F. Vallentin for $m_1(\mathbb{R}^n)$ (2010).
Lovász theta number

The theta number $\vartheta(G)$ (L. Lovász, 1979) satisfies the Sandwich Theorem:

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G})$$

It is the optimal value of a semidefinite program

Idea: if $S$ is an independence set of $G$, consider the matrix

$$B_S(x, y) := \mathbf{1}_S(x) \mathbf{1}_S(y)/|S|.$$ 

$B_S \succeq 0$, $B_S(x, y) = 0$ if $xy \in E$, $|S| = \sum_{(x,y) \in V^2} B_S(x, y)$. 
\( \vartheta(G) \)

- Defined by:

\[
\vartheta(G) = \max \left\{ \sum_{(x,y) \in V^2} B(x, y) : B \in \mathbb{R}^{V \times V}, B \succeq 0, \right. \\
\left. \sum_{x \in V} B(x, x) = 1, \right. \\
B(x, y) = 0 \quad xy \in E \right\}
\]

- Proof of \( \alpha(G) \leq \vartheta(G) \): Let \( S \) be an independent set. \( B_S(x, y) = \mathbf{1}_S(x) \mathbf{1}_S(y) / |S| \) satisfies the constraints of the above SDP. Thus

\[
\sum_{(x,y) \in V^2} B_S(x, y) = |S| \leq \vartheta(G).
\]
Over $\mathbb{R}^n$: take $B(x, y)$ continuous, positive definite, i.e. for all $k$, for all $x_1, \ldots, x_k \in \mathbb{R}^n$, $(B(x_i, x_j))_{1 \leq i, j \leq k} \succeq 0$.

Assume $B$ is translation invariant: $B(x, y) = f(x - y)$ (the graph itself is invariant by translation).

Replace $\sum_{(x,y) \in V^2} B(x, y)$ by

$$\delta(f) := \limsup_{r \to +\infty} \frac{1}{\text{vol}(B_n(r))} \int_{B_n(r)} f(z)dz.$$
leads to:

\[ \vartheta(\mathbb{R}^n) := \sup \left\{ \delta(f) : f \in C_b(\mathbb{R}^n), f \succeq 0, f(0) = 1, f(x) = 0 \quad \|x\| = 1 \right\} \]

**Theorem**  
*(Oliveira Vallentin 2010)*

\[ m_1(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n) \]
The computation of $\mathcal{V}(\mathbb{R}^n)$

- Bochner characterization of positive definite functions:

$$f \in C(\mathbb{R}^n), f \succeq 0 \iff f(x) = \int_{\mathbb{R}^n} e^{ix \cdot y} d\mu(y), \mu \geq 0.$$ 

- $f$ can be assumed to be radial i.e. invariant under $O(\mathbb{R}^n)$:

$$f(x) = \int_0^{+\infty} \Omega_n(t\|x\|)d\alpha(t), \alpha \geq 0.$$ 

where

$$\Omega_n(t) = \Gamma(n/2)(2/t)^{(n/2-1)}J_{n/2-1}(t).$$

- Then take the dual program.
The computation of $\vartheta(\mathbb{R}^n)$

- Leads to:

$$\vartheta(\mathbb{R}^n) = \inf \{ z_0 : \quad z_0 + z_1 \geq 1 $$
$$z_0 + z_1 \Omega_n(t) \geq 0 \quad \text{for all } t > 0 \}$$

- Explicitly solvable. For $n = 4$, graphs of $\Omega_4(t)$ and of the optimal function $f_4^*(t) = z_0^* + z_1^* \Omega_4(t)$:

The minimum of $\Omega_n(t)$ is reached at $j_{n/2,1}$ the first zero of $J_{n/2}$. 
The computation of $\mathcal{V}(\mathbb{R}^n)$

- We obtain

$$f_n^*(t) = \frac{\Omega_n(t) - \Omega_n(j_{n/2}, 1)}{1 - \Omega_n(j_{n/2}, 1)}$$

$$\mathcal{V}(\mathbb{R}^n) = \frac{-\Omega_n(j_{n/2}, 1)}{1 - \Omega_n(j_{n/2}, 1)}.$$

- Resulting upper bound for $m_1(\mathbb{R}^n)$ (OV 2010):

$$m_1(\mathbb{R}^n) \leq \mathcal{V}(\mathbb{R}^n) = \frac{-\Omega_n(j_{n/2}, 1)}{1 - \Omega_n(j_{n/2}, 1)}.$$

- Decreases exponentially but not as fast as Frankl Wilson Raigorodskii bound ($1.165^{-n}$ instead of $1.239^{-n}$). A weaker bound, but with the same asymptotic, was obtained in BNOV 2009 through $m(S^{n-1}, \theta)$. 
To summarize, we have seen two essentially different bounds:

\[ m_1(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|} \quad \text{with FW graphs and lin. alg. bound} \]

\[ m_1(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n) \quad \text{morally encodes } \vartheta(G) \text{ for every } G \hookrightarrow \mathbb{R}^n \]

The former is the best asymptotic while the later improves the previous bounds in the range \(3 \leq n \leq 24\).

It is possible to combine the two methods, i.e. to insert the constraint relative to a finite graph \(G\) inside \(\vartheta(\mathbb{R}^n)\). Joint work (in progress) with F. Oliveira and F. Vallentin.
Let $G \hookrightarrow \mathbb{R}^n$, for $x_i \in V$, let $r_i := \|x_i\|$. 

$$\vartheta_G(\mathbb{R}^n) := \inf \{ z_0 + z_2 \frac{\alpha(G)}{|V|} : z_2 \geq 0 $$
$$z_0 + z_1 + z_2 \geq 1 $$
$$z_0 + z_1 \Omega_n(t) + z_2(\frac{1}{|V|} \sum_{i=1}^{|V|} \Omega_n(r_i t)) \geq 0 $$
for all $t > 0 \}$. 

Theorem

$$m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n)$$
Sketch of proof

- \( \vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n) \) is obvious: take \( z_2 = 0 \).
- Sketch proof of \( m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n) \): let \( S \) a measurable set avoiding 1. Let
  \[ f_S(x) := \frac{\delta(1_{S-x} 1_S)}{\delta(S)}. \]
  \( f_S \) is continuous bounded, \( f_S \geq 0 \), \( f_S(0) = 1 \), \( f_S(x) = 0 \) if \( \|x\| = 1 \). Moreover \( \delta(f_S) = \delta(S) \).
- Thus \( f_S \) is feasible for \( \vartheta(\mathbb{R}^n) \), which proves that \( \delta(S) \leq \vartheta(\mathbb{R}^n) \).
Sketch of proof

- If $V = \{x_1, \ldots, x_M\}$, for all $y \in \mathbb{R}^n$,

$$\sum_{i=1}^{M} 1_{S-x_i}(y) \leq \alpha(G).$$

- Leads to the extra condition:

$$\sum_{i=1}^{M} f_S(x_i) \leq \alpha(G).$$

- Design a linear program, apply Bochner theorem, symmetrize by $O(\mathbb{R}^n)$, take the dual.
Bad news: cannot be solved explicitly (we don’t know how to)
Challenge: to compute good feasible functions.
First method: to sample an interval $[0, M]$, solve a finite LP, then adjust the optimal solution ($OV, G = simplex$).

Figure: $f^*_4(t)$ (blue) and $f^*_{4, G}(t)$ (red) for $G = simplex$
Observation: the optimal has a zero at $y > \frac{j_{n/2,1}}{2}$.

Idea: to parametrize $f = z_0 + z_1 \Omega_n(t) + z_2 \Omega_n(rt)$ with $y$:

$f(y) = f'(y) = 0$, $f(0) = 1$ determines $f$.

We solve for:

$$\begin{cases} 
    z_0 + z_1 + z_2 = 1 \\
    z_0 + z_1 \Omega_n(y) + z_2 \Omega_n(ry) = 0 \\
    z_1 \Omega'_n(y) + rz_2 \Omega'_n(ry) = 0
\end{cases}$$

Then, starting with $y = j_{n/2,1}$, we move $y$ to the right until $f_y(t) := z_0(y) + z_1(y)\Omega_n(t) + z_2(y)\Omega_n(rt)$ takes negative values.
Numerical results: upper bounds for $m_1(\mathbb{R}^n)$

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Numerical results: lower bounds for $\chi_m(\mathbb{R}^n)$

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Questions, comments

- Exponential behavior of $\vartheta_{FW}(\mathbb{R}^n)$?
- Further improvements for small dimensions: change the graph, consider several graphs. For $n = 2$, several triangles lead to 0.268412 (OV); several Moser spindles to 0.262387 (F. Oliveira 2011).
- Can we reach $m_1(\mathbb{R}^2) < 0.25$? (conjectured by Erdös; would give another proof of $\chi_{m}(\mathbb{R}^2) \geq 5$).
- Applies to other spaces, e.g. $m(S^{n-1}, \theta)$ (BNOV 2009).
- In turn, a bound for $m_1(S(0, r))$ can replace a finite graph $G$ in $\vartheta_G(\mathbb{R}^n)$.
- The Lovász theta method was successfully adapted to $\mathbb{R}^n$. What about the linear algebra method (Gil Kalai)?