Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Exercise sheet 11** (14.01.2026)

(11.1) Exercise: Gluing sheaves.
Let V be a topological space, and let $\{V_i \subseteq V; i \in \mathcal{I}\}$ be an open covering, where $\mathcal{I}$ is an index set. For all $i \in \mathcal{I}$ let $\mathcal{F}_i$ be a sheaf on V_i, with values in a fixed category, such that there are isomorphisms $\varphi_{ij} \colon (\mathcal{F}_i)|_{V_i \cap V_j} \Rightarrow (\mathcal{F}_j)|_{V_i \cap V_j}$ fulfilling $\varphi_{ij}\varphi_{jk} = \varphi_{ik} \colon (\mathcal{F}_i)|_{V_i \cap V_j \cap V_k} \Rightarrow (\mathcal{F}_k)|_{V_i \cap V_j \cap V_k}$, for all $i, j, k \in \mathcal{I}$.

a) Show that we have $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i} \colon \mathcal{F}_i \Rightarrow \mathcal{F}_i$, for all $i \in \mathcal{I}$.

b) Show that there is a sheaf $\mathcal{F}$ on V, together with isomorphisms $\psi_i \colon \mathcal{F}_i \Rightarrow \mathcal{F}|_{V_i}$, such that $\psi_i = \varphi_{ij}\psi_j \colon (\mathcal{F}_i)|_{V_i \cap V_j} \Rightarrow \mathcal{F}|_{V_i \cap V_j}$, for all $i, j \in \mathcal{I}$. Moreover, show that $\mathcal{F}$ and the ψ_i are uniquely defined up to unique isomorphism.

(11.2) Exercise: Classification of binary quadrics.
a) Let L be an algebraically closed field, such that $\mathrm{char}(L) \neq 2$, let $A := L[X, Y]$, let $f \in A$ be irreducible of degree 2, and let $\mathbf{Q} := \mathbf{V}_L(f)$ be the associated **binary quadric**. Show that either $L[\mathbf{Q}] \cong L[\mathbf{V}]$ or $L[\mathbf{Q}] \cong L[\mathbf{W}]$, where $\mathbf{V} := \mathbf{V}(Y - X^2)$ and $\mathbf{W} := \mathbf{V}(XY - 1)$. Find a criterion on the coefficients of f to decide which case occurs. Conclude that $\mathbf{Q}$ is isomorphic to L or $L \setminus \{0\}$.

b) Let $A^\sharp := L[X, Y, Z]$. Show that the projective closures $\overline{\mathbf{V}} \subseteq \mathbf{P}^2$ and $\overline{\mathbf{W}} \subseteq \mathbf{P}^2$ are both isomorphic to $\mathbf{V}^\sharp(XZ - Y^2) \subseteq \mathbf{P}^2$.

(11.3) Exercise: Morphisms of projective varieties.
Let L be an algebraically closed field, and let $p \in \mathbf{P}^2$.
a) Show that the set of lines in $\mathbf{P}^2$ through p can be identified with $\mathbf{P}^1$.
b) We consider the **conic** $\mathbf{V} := \mathbf{V}^\sharp(XZ - Y^2) \subseteq \mathbf{P}^2$, where $L[X, Y, Z]$ is the homogeneous coordinate algebra of $\mathbf{P}^2$. Show that $\mathbf{V}$ is irreducible
c) Let $p := [0\colon 0\colon 1] \in \mathbf{V}$. Show that any line in $\mathbf{P}^2$ through p intersects $\mathbf{V}$ in precisely two points, where for the 'tangent' at $\mathbf{V}$ the point p is counted twice.
d) Show that this yields an isomorphism of projective varieties $\mathbf{P}^1 \to \mathbf{V}$.

(11.4) Exercise: Veronese embeddings again.
Let $K \subseteq L$ be a field extension, where L algebraically closed, let $\varphi \colon \mathbf{P}^n \to \mathbf{P}^N$ be the d-fold Veronese embedding, where $N := \binom{n+d}{n} - 1$.
a) Show that φ induces an isomorphism of varieties $\mathbf{P}^n \cong \varphi(\mathbf{P}^n)$.
b) Let $\varphi(\mathbf{P}^1) \subseteq \mathbf{P}^2$ be the image of the 2-fold Veronese embedding of $\mathbf{P}^1$. Show that for the homogeneous coordinate algebras we have $K[\mathbf{P}^1] \ncong K[\varphi(\mathbf{P}^1)]$.

(11.5) Exercise: Quasi-affine varieties.
Show that the affine line is not isomorphic as a variety to any proper open subset of itself.