Übungen zur Algebraischen Zahlentheorie (WS 2023)

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(6.1) Exercise: Diophantine equations.
Show the **Ramanujan-Nagell Theorem**, saying that $n \in \pm\{1, 3, 5, 11, 181\}$ are the only integers such that $n^2 + 7$ is a 2-power.

Hint. Use the ring $\mathbb{Z}[\alpha]$, where $\alpha := \frac{1}{2}(1+\sqrt{-7})$, which is known to be factorial, and recall that $N(\alpha) = 2$.

(6.2) Exercise: Factorisation of ideals.
For $d \in \mathbb{Z}\setminus\{0, 1\}$ square-free let $\mathcal{O}_d$ and $\widehat{\mathcal{O}}_d$ be the rings of integers of $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{d})$, respectively. Determine the factorisation of the following ideals:

a) i) $(10) \trianglelefteq \mathcal{O}_{15}$, **ii)** $(6) \trianglelefteq \mathcal{O}_{30}$.

b) i) $(6) \trianglelefteq \mathcal{O}_{-6}$, **ii)** $(14) \trianglelefteq \mathcal{O}_{-10}$, **iii)** $(18) \trianglelefteq \mathcal{O}_{-17}$, **iii)** $(30) \trianglelefteq \mathcal{O}_{-29}$.

c) i) $(6) \trianglelefteq \widehat{\mathcal{O}}_{-3}$, **ii)** $(14) \trianglelefteq \widehat{\mathcal{O}}_{-5}$

(6.3) Exercise: Ramification in cubic fields.
Let $\alpha := \sqrt[3]{2}$ and $K := \mathbb{Q}(\alpha)$.
a) Determine the embeddings of K into $\mathbb{C}$, show that its ring of integers equals $\mathcal{O} = \mathbb{Z}[\alpha]$, and compute its discriminant.
b) Determine the factorisation of the ideal $(5) \trianglelefteq \mathcal{O}$, and the associated ramification indices and inertial degrees.

(6.4) Exercise: Ramification in cubic fields.
Let $\alpha \in \mathbb{R}$ such that $\alpha^3 = \alpha + 1$, and let $K := \mathbb{Q}(\alpha)$.
a) Show that $\mathcal{O} := \mathbb{Z}[\alpha]$ is the ring of integers of K, and determine $\mathrm{disc}(\mathcal{O})$.
b) Determine the factorisation of the ideal $(23) \trianglelefteq \mathcal{O}$, and the associated ramification indices and inertial degrees.