Ferienübung zur Algebraischen Zahlentheorie (WS 2023)

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(F.1) Exercise: Fermat equation.
Show that for $n = 4$ there are no non-trivial integral solutions of the Fermat equation $X^n + Y^n = Z^n$.

Hint. Consider a primitive solution $[x, y, u] \in \mathbb{N}^3$ of the equation $X^4+Y^4 = U^2$, where u is minimal, and use the classification of primitive Pythagorean triples.

(F.2) Exercise: Ramification in function fields.
Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We consider the plane curve $\{[t^2, t] \in \mathbb{F}^2; t \in \mathbb{F}\}$, having affine coordinate algebra $S := \mathbb{F}[X, Y]/(Y^2 - X)$. Letting $R := \mathbb{F}[X]$ and $K := \mathrm{Q}(R)$, the projection morphism onto the first coordinate gives rise to the natural $\mathbb{F}$-algebra homomorphism $\pi\colon R \to S\colon X \mapsto \overline{X}$.

a) Show the following: The ring S is a univariate polynomial algebra; let $L := \mathrm{Q}(S)$. The map π is injective; hence we may consider R as a subalgebra of S.

The field extension $K \subseteq L$ is Galois; determine the degree $[L\colon K]$. The ring extensiom $R \subseteq S$ is finite, S is the integral closure of R in L, and S is a free R-module. Determine the trace R-bilinear form of S, and compute the discriminant $\mathrm{disc}_R(S) \in R/(\mathbb{F}^*)^2$ of S.

b) Determine the sets $\mathcal{P}_R$ and $\mathcal{P}_S$ of non-zero prime ideals of R and S, respectively, and show the following:

Both sets consist of maximal ideals. The sets $\mathcal{P}_S(\mathfrak{p}) := \{\mathfrak{q} \in \mathcal{P}_S; \mathfrak{q} \cap R = \mathfrak{p}\}$ for $\mathfrak{p} \in \mathcal{P}_R$, are non-empty and form a partition of $\mathcal{P}_S$. Actually, $\mathcal{P}_S(\mathfrak{p})$ consists of a single orbit under $\mathrm{Aut}_K(L)$, and we have $\mathfrak{p}S = (\prod_{\mathfrak{q}\in\mathcal{P}_S(\mathfrak{p})} \mathfrak{q})^{e_\mathfrak{p}} \lhd S$, for some $e_\mathfrak{p} \in \mathbb{N}$. What is the geometrical interpretation of the ramification index $e_\mathfrak{p}$? How does ramification relate to the discriminant $\mathrm{disc}_R(S)$?

Letting $\mathbb{E} := S/\mathfrak{q}$, then $\mathbb{E}$ is independent of the choice of $\mathfrak{q} \in \mathcal{P}_S(\mathfrak{p})$; determine the embedding $\mathbb{F} \subseteq \mathbb{E}$, and the degree $f_\mathfrak{p} := [\mathbb{E}\colon \mathbb{F}] \in \mathbb{N}$. We have $|\mathcal{P}_S(\mathfrak{p})| \cdot e_\mathfrak{p} \cdot f_\mathfrak{p} = [L\colon K]$. What is the geometrical interpretation of the inertia degree $f_\mathfrak{p}$? Determine the associated decomposition and inertia groups.

Hint for b). For $\mathbb{F} = \mathbb{C}$ distinguish the cases $x = 0$ and $x \neq 0$, while for $\mathbb{F} = \mathbb{R}$ distinguish the cases $x = 0$ and $x > 0$ and $x < 0$.

(F.3) Exercise: Carmen de Hastingae Proelio.
All historians know that there is a great deal of mystery and uncertainty concerning the details of the ever-memorable battle [near Hastings] *on that fatal day, October 14, 1066. Here is the passage in question:*

The men of Harold stood well together, as their wont was, and formed sixty and one squares, with a like number of men in every square thereof, and woe to the hardy Norman who ventured to enter their redoubts; for a single blow of a Saxon war-hatchet would break his lance and cut through his coat of mail... When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle-cries 'Ut!', 'Olicrosse!', 'Godemitè!'.

What is the smallest possible number of men there could have been?

(F.4) Exercise: Archimedes's Problema Bovinum.
If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, a third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all of the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

But come, understand also all these conditions regarding the cattle of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

Hint. The first part amounts to solving a system of diophantine linear equations, the second part leads to a Pell equation; anyway you better use GAP.