Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 25** (12.01.2026)

(25.1) Example: The quadric hypersurface. We show that a regular function on a quasi-affine variety is not necessarily globally rational:

Let $K = L = \mathbb{C}$, and let $f := WX - YZ \in A := \mathbb{C}[W, X, Y, Z]$. Then, since f is homogeneous, it is immediate that f is irreducible. We consider the irreducible affine hypersurface $\mathbf{V} := \mathbf{V}(f) \subseteq \mathbb{C}^4$, where $\mathbb{C}[\mathbf{V}] = A/\langle f \rangle$ is a graded domain.

Since f has degree 2, we conclude that $\{W, X, Y, Z\} \subseteq \mathbb{C}[\mathbf{V}]_1$ is a $\mathbb{C}$-basis. In particular, the elements W, X, Y, Z are irreducible and pairwise non-associated. This implies that none of these elements is prime: It suffices to consider X; we have $X \mid YZ \in \mathbb{C}[\mathbf{V}]$, but $X \nmid Y$ and $X \nmid Z$. Thus $\mathbb{C}[\mathbf{V}]$ is not factorial.

Next we show that the (necessarily homogeneous) divisors of $Y^m \in \mathbb{C}[\mathbf{V}]$, where $m \in \mathbb{N}_0$, up to associates are Y^k for $k \leq m$, up to associates:

First, since f is binomial, we infer that $\{Y^i; i \geq 0\} \cdot \mathcal{B} \subseteq \mathbb{C}[\mathbf{V}]$, where $\mathcal{B} := \{Z^i; i \geq 0\} \cdot \big(\{1\} \,\dot\cup\, \{X^i; i \geq 1\} \,\dot\cup\, \{W^i; i \geq 1\}\big)$, is a $\mathbb{C}$-basis consisting of monomials. The product of two basis elements is a basis element again, except that for $a, b, c, d \geq 0$ and $i, j \geq 1$ we have $(Y^a Z^b X^i)(Y^c Z^d W^j) = Y^{a+c+j} Z^{b+d+j} X^{i-j}$ if $i \geq j$, and $(Y^a Z^b X^i)(Y^c Z^d W^j) = Y^{a+c+i} Z^{b+d+i} W^{j-i}$ if $i \leq j$.

Now we proceed by induction on $m \geq 0$, where for $m = 0$ the assertion is trivial; let $m \geq 1$: Let $Y^m = g'g''$, where $g' = \sum_{j=0}^{k} p'_{k-j} Y^j$ and $g'' = \sum_{j=0}^{l} p''_{l-j} Y^j$ are homogeneous of degree k and l, respectively, such that $k + l = m$, and $p'_j, p''_j \in \langle \mathcal{B} \rangle_{\mathbb{C}}$ are homogeneous of degree j. From the above multiplication rules, since $k + l \geq 1$, we get $p'_k p''_l = 0$. Since $\mathbb{C}[\mathbf{V}]$ is a domain, this entails $p'_k = 0$, say. Writing $g' = g''' \cdot Y$ we get $g''' g'' = Y^{m-1}$, and apply induction. ♯

Now let $U := D_W \cup D_Y \subseteq \mathbf{V}$ open, and let $\varphi \in \mathcal{O}_{\mathbf{V}}(U)$ such that $\varphi|_{D_W} = \frac{Z}{W}$ and $\varphi|_{D_Y} = \frac{X}{Y}$, where since $\frac{Z}{W}|_{D_{WY}} = \frac{X}{Y}|_{D_{WY}}$ the function φ is well-defined.

Assume that $\varphi = \frac{h}{g}$ on U, where $0 \neq g, h \in \mathbb{C}[\mathbf{V}]$; hence we have $\mathbf{V}(g) \subseteq \mathbf{V} \setminus U = \mathbf{V}_{\mathbf{V}}(W, Y)$. Since $f \in \langle W, Y \rangle_A \trianglelefteq A$, we have $\mathbb{C}[\mathbf{V}]/\langle W, Y \rangle \cong A/\langle W, Y \rangle_A \cong \mathbb{C}[X, Z]$, hence $\langle W, Y \rangle \trianglelefteq \mathbb{C}[\mathbf{V}]$ is prime. Thus we get $\langle W, Y \rangle \subseteq \sqrt{\langle g \rangle} \trianglelefteq \mathbb{C}[\mathbf{V}]$, hence there is $l \in \mathbb{N}$ such that $g \mid Y^l \in \mathbb{C}[\mathbf{V}]$.

Hence we may assume that $g = Y^k$, for some $k \in \mathbb{N}_0$. Then considering $v := [1, 0, 0, 1] \in D_W \setminus D_Y \subseteq U \subseteq \mathbf{V}$ yields $0^k = Y^k(v) = g(v) \neq 0$, thus $k = 0$, that is $g = 1$. By continuity, this entails $h = \frac{X}{Y} \in \mathbb{C}(\mathbf{V})$. Thus we get $h \cdot Y = X \in \mathbb{C}[\mathbf{V}]$, implying that X and Y are associated, a contradiction.

(25.2) Example. We show that a quasi-affine variety is not necessarily affine:

Letting $K = L = \mathbb{C}$, we consider the affine space $\mathbf{V} := \mathbb{C}^2$, having coordinate algebra $A := \mathbb{C}[X, Y]$, and the open subset $U := D_X \cup D_Y = \mathbf{V} \setminus \{[0, 0]\} \subseteq \mathbf{V}$, having structure sheaf $\mathcal{O}_U = \mathcal{O}_{\mathbf{V}}|_U$. We determine $\Gamma(\mathcal{O}_U) = \mathcal{O}_{\mathbf{V}}(U)$:

Letting $\varphi \in \mathcal{O}_{\mathbf{V}}(U)$, there are $f, g \in A$ and $k, l \in \mathbb{N}_0$ such that $\varphi|_{D_X} = \frac{f}{X^k} \in A_X$ and $\varphi|_{D_Y} = \frac{g}{Y^l} \in A_Y$; recall that by continuity $\mathrm{Q}(A)$ can be identified with the set of regular functions it induces. Thus on $D_X \cap D_Y = D_{XY}$ we have $\frac{f}{X^k} = \frac{g}{Y^l} \in A_{XY}$, from which we get $fY^l = gX^k \in A$. Since A is factorial, where X and Y are irreducible and non-associated, we infer that $X^k \mid f$ and $Y^l \mid g$, hence $\varphi \in A$. Conversely, $\rho^{\mathbf{V}}_{D_{XY}} \colon A \to A_{XY}$ is injective, hence so is $\rho^{\mathbf{V}}_U \colon A \to \mathcal{O}_{\mathbf{V}}(U)$. This implies $\mathcal{O}_{\mathbf{V}}(U) = A$ and $\rho^{\mathbf{V}}_U = \mathrm{id}_A$. $\sharp$

Let $\iota^{\mathbf{V}}_U$ be the natural inclusion map; recall $(\iota^{\mathbf{V}}_U)^* = \rho^{\mathbf{V}}_U$. Then $\iota^{\mathbf{V}}_U$ is a morphism of varieties: It is continuous, and for any $W \subseteq \mathbf{V}$ open we have

$$(\iota^{\mathbf{V}}_U)^*_W(\mathcal{O}_{\mathbf{V}}(W)) = \rho^W_{W\cap U \subseteq \mathbf{V}}(\mathcal{O}_{\mathbf{V}}(W)) \subseteq \mathcal{O}_{\mathbf{V}}(W \cap U) = \mathcal{O}_U((\iota^{\mathbf{V}}_U)^{-1}(W)).$$

Now assume that U is affine. Then $\iota^{\mathbf{V}}_U$ is a morphism of affine varieties, that is a regular map. The associated comorphism induces the identity $\mathrm{id}_A = \rho^{\mathbf{V}}_U \colon \Gamma(\mathcal{O}_{\mathbf{V}}) = A \to A = \Gamma(\mathcal{O}_U)$ on global sections. The latter is an isomorphism of K-algebras, so that $\iota^{\mathbf{V}}_U$ is an isomorphism of varieties, hence is bijective, a contradiction. This shows that U together with $\mathcal{O}_U$ is not an affine variety.
