Algebraic Geometry (WS 2025)

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(26.1) Restricted sheaves again. Let V be a topological space, and let $\mathcal{O}_V$ be a presheaf of functions on V, whose restriction maps are given by restriction of functions. The case of $W \subseteq V$ being open is of particular interest: In this case, any open subset $U \subseteq W$ is open in V as well, and $(\iota_W^V)_U^* \colon \mathcal{O}_V(U) \to \mathcal{O}_V|_W(U) \colon f \mapsto f|_U = f$ is the identity map, hence is injective; thus the morphism of presheaves $(\iota_W^V)^* \colon \mathcal{O}_V \Rightarrow \mathcal{O}_V|_W$ is called **injective**.

We define the presheaf $\mathcal{O}_W$ on W by letting $\mathcal{O}_W(U) := \mathcal{O}_V(U)$, for any $U \subseteq W$ open, where the restriction maps of $\mathcal{O}_W$ are inherited from $\mathcal{O}_V$. Thus we get an injective morphism of presheaves $\sigma_{V|W} \colon \mathcal{O}_W \Rightarrow \mathcal{O}_V|_W$. In particular, $\sigma_{V|V} \colon \mathcal{O}_V \Rightarrow \mathcal{O}_V|_V$ is injective, saying that $\mathcal{O}_V$ is a **subpresheaf** of $\mathcal{O}_V|_V$, where the sheaf $\mathcal{O}_V|_V$ is called the **sheafification** of $\mathcal{O}_V$. (The sheafification fulfills a universal property, which we do not discuss here.)

If $\mathcal{O}_V$ already is a sheaf, we show that $\sigma_{V|W} \colon \mathcal{O}_W \Rightarrow \mathcal{O}_V|_W$ is **surjective** as well, that is $(\iota_W^V)_U^* \colon \mathcal{O}_V(U) \to \mathcal{O}_V|_W(U)$ is surjective, for all $U \subseteq W$ open:

Let $f \in \mathcal{O}_V|_W(U)$. Then for all $v \in U$ let $v \in U_v \subseteq V$ be open, and let $f_v \in \mathcal{O}_V(U_v)$, such that $U_v \cap W \subseteq U$ and $f_v|_{U_v \cap W} = f|_{U_v \cap W}$. Since W is open, we may assume that $U_v \subseteq W$, or equivalently $U_v \subseteq U$, and thus $f_v = f|_{U_v}$. Since $\{U_v; v \in U\}$ is an open covering of U, the sheaf properties imply $f \in \mathcal{O}_V(U)$. $\sharp$

Thus in this case we have $\sigma_{V|W}(U) = \mathrm{id}_{\mathcal{O}_W(U)}$, for all $U \subseteq W$ open, so that we recover the earlier definition of restricting a sheaf to an open subset. In particular, sheafification does not change a given sheaf.

(26.2) Subprevarieties. We return to the earlier setting, and let $(V, \mathcal{O}_V)$ be a prevariety. We show that open and irreducible closed subsets of V, respectively, carry the structure of a prevariety again:

Proposition. Let $U \subseteq V$ be open. Then $(U, \mathcal{O}_V|_U)$ is a prevariety again, being called an **open subprevariety**.

Proof. We may assume that $U \neq \emptyset$. Since V is irreducible, U is dense, hence is irreducible as well, thus is connected. Moreover, since V is quasi-compact, so is U. Finally, since the affine open subsets form a basis of the topology on V, we conclude that U is covered by affine open subsets. $\sharp$

To deal with the case of closed subsets, let first $\mathbf{V}$ be irreducible affine with structure sheaf $\mathcal{O}_{\mathbf{V}}$, and let $\mathbf{W} \subseteq \mathbf{V}$ be closed and irreducible. Then $\mathbf{W}$ carries the structure of an irreducible affine variety with structure sheaf $\mathcal{O}_{\mathbf{W}}$, and the structure of a space with functions with respect to the sheaf $\mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}$. The natural inclusion map $\iota_{\mathbf{W}}^{\mathbf{V}} \colon \mathbf{W} \to \mathbf{V}$, which is regular, induces a comorphism $(\iota_{\mathbf{W}}^{\mathbf{V}})^* \colon \mathcal{O}_{\mathbf{V}} \Rightarrow \mathcal{O}_{\mathbf{W}}$, as well as a comorphism $(\iota_{\mathbf{W}}^{\mathbf{V}})^* \colon \mathcal{O}_{\mathbf{V}} \Rightarrow \mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}$ anyway.

Proposition. We have $\mathcal{O}_{\mathbf{V}}|_{\mathbf{W}} = \mathcal{O}_{\mathbf{W}}$.

Proof. Letting $U \subseteq \mathbf{W}$ open, we show that $\mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}(U) = \mathcal{O}_{\mathbf{W}}(U)$:

Let first $f \in \mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}(U)$, for $v \in U$ let $v \in U_v \subseteq \mathbf{V}$ be open, and let $f_v \in \mathcal{O}_{\mathbf{V}}(U_v)$, such that $U_v \cap \mathbf{W} \subseteq U$ and $f_v|_{U_v \cap \mathbf{W}} = f|_{U_v \cap \mathbf{W}}$. Thus we have $f|_{U_v \cap \mathbf{W}} \in \mathcal{O}_{\mathbf{W}}(U_v \cap \mathbf{W})$, by the sheaf properties implying $f \in \mathcal{O}_{\mathbf{W}}(U)$.

Conversely, again by the sheaf properties, it suffices to consider principal open subsets $D_g \subseteq \mathbf{W}$, where $0 \neq g \in K[\mathbf{W}]$. Letting $I := \mathbf{I}_{\mathbf{V}}(\mathbf{W}) \trianglelefteq K[\mathbf{V}]$, we have the natural epimorphism $(\iota_{\mathbf{W}}^{\mathbf{V}})^*_{\mathbf{V}} \colon K[\mathbf{V}] \to K[\mathbf{V}]/I = K[\mathbf{W}]$. Thus we get the natural epimorphism $K[\mathbf{V}]_{K[\mathbf{V}] \setminus I} \to \mathrm{Q}(K[\mathbf{W}]) = K(\mathbf{W})$; note that I being prime, $K[\mathbf{V}] \setminus I$ is multiplicatively closed. Thus any $f = \frac{h}{g^k} \in \mathcal{O}_{\mathbf{W}}(D_g) = K[\mathbf{W}]_g \subseteq K(\mathbf{W})$ lifts to an element $\widehat{f} = \frac{\widehat{h}}{\widehat{g}^k} \in K[\mathbf{V}]_{\widehat{g}} \subseteq K[\mathbf{V}]_{K[\mathbf{V}] \setminus I}$. Then we have $D(\widehat{g}) \cap \mathbf{W} = D(g)$ and $\widehat{f}|_{D(\widehat{g}) \cap \mathbf{W}} = f$, thus $f \in \mathcal{O}_{\mathbf{V}}|_{\mathbf{W}}(D_g)$. ♯

Corollary. Let $W \subseteq V$ be closed and irreducible. Then $(W, \mathcal{O}_V|_W)$ is a prevariety again, being called a **closed subprevariety**.

Proof. Since W is connected, it only remains to be shown that $(W, \mathcal{O}_V|_W)$ has a finite affine open covering: To this end, let $\{V_i; i \in \mathcal{I}\}$ be an affine open covering of V, where $\mathcal{I}$ is a finite index set. Then $\{V_i \cap W; i \in \mathcal{I}\}$ is a finite open covering of W; note that $V_i \cap W$ is empty or irreducible. Then since $V_i \cap W \subseteq W$ and $V_i \subseteq V$ are open, we have $(\mathcal{O}_V|_W)|_{V_i \cap W} = \mathcal{O}_V|_{V_i \cap W} = (\mathcal{O}_V|_{V_i})|_{V_i \cap W}$. Since $\mathcal{O}_V|_{V_i} = \mathcal{O}_{V_i}$ is affine, and $V_i \cap W \subseteq V_i$ is closed, we conclude that $(\mathcal{O}_V|_{V_i})|_{V_i \cap W} = \mathcal{O}_{V_i}|_{V_i \cap W} = \mathcal{O}_{V_i \cap W}$ is affine. ♯
