Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 27** (14.01.2026)

(27.1) Proposition. Let $(V, \mathcal{O}_V)$ and $(W, \mathcal{O}_W)$ be prevarieties, let $\{V_i; i \in \mathcal{I}\}$ be an open covering of V, let $\{W_i; i \in \mathcal{I}\}$ be an affine open covering of W, where $\mathcal{I}$ is an index set, and let $\varphi \colon V \to W$ be a map such that $\varphi(V_i) \subseteq W_i$ and $(\varphi|_{V_i})^*(\mathcal{O}_W(W_i)) \subseteq \mathcal{O}_V(V_i)$, for all $i \in \mathcal{I}$. Then φ is a morphism.

Proof. If $U \subseteq V_i$ is open, then from $(\varphi|_U)^*(f) = (\varphi|_{V_i})^*(f)|_U$, for any $f \colon W_i \to L$, we get $(\varphi|_U)^*(\mathcal{O}_W(W_i)) \subseteq \mathcal{O}_V(U)$. Hence, since the affine open subsets form a basis of the topology on V, we may assume that the V_i are affine open.

Then, abbreviating $\varphi_i := \varphi|_{V_i}$, since $\mathcal{O}_V(V_i) = \Gamma(V_i)$ and $\mathcal{O}_W(W_i) = \Gamma(W_i)$, we have $(\varphi_i)^*(\Gamma(W_i)) \subseteq \Gamma(V_i)$. Thus, V_i and W_i being affine, we conclude that φ_i is a morphism. In particular, φ_i is continuous, thus φ is continuous as well.

For $U \subseteq W$ open, the set $\{U \cap W_i; i \in \mathcal{I}\}$ is an open covering of U, and we have $\varphi_i^{-1}(U \cap W_i) = \varphi^{-1}(U \cap W_i) \cap V_i = \varphi^{-1}(U) \cap V_i$. Then for $f \in \mathcal{O}_W(U)$ we have $f|_{U \cap W_i} \in \mathcal{O}_W(U \cap W_i) = \mathcal{O}_{W_i}(U \cap W_i)$, hence we get

$$\varphi^*(f)|_{\varphi^{-1}(U) \cap V_i} = (\varphi_i)^*(f|_{U \cap W_i}) \in \mathcal{O}_{V_i}(\varphi_i^{-1}(U \cap W_i)) = \mathcal{O}_V(\varphi^{-1}(U) \cap V_i).$$

Since $\{\varphi^{-1}(U) \cap V_i; i \in \mathcal{I}\}$ is an open covering of $\varphi^{-1}(U)$, by the sheaf properties from this we infer that $\varphi^*(f) \in \mathcal{O}_V(\varphi^{-1}(U))$. ♯

(27.2) Example: Affine line with an additional point. We keep the above notation. Let $U_1 \cong L \cong U_2$ be two disjoint copies of the affine line, having coordinate algebras $K[U_1] = K[S]$ and $K[U_2] = K[T]$, and let $U_1' := (U_1)_S \cong L \setminus \{0\}$ and $U_2' := (U_2)_T \cong L \setminus \{0\}$. Then U_1' and U_2' are affine varieties such that $K[U_1'] = K[S]_S = K[S^{\pm 1}]$ and $K[U_2'] = K[T]_T = K[T^{\pm 1}]$.

We 'glue' U_1 and U_2 along an identification $\varphi \colon U_1' \to U_2'$ as affine varieties. To do so, it suffices to pick any isomorphism of K-algebras $\varphi^* \colon K[T^{\pm 1}] \to K[S^{\pm 1}]$.

Let $V := U_1 \coprod_\varphi U_2$ be the **fibre sum** of U_1 and U_2 along φ; that is we have embeddings $\iota_i \colon U_i \to V$, for $i \in \{1, 2\}$, such that letting $V_i := \iota_i(U_i)$ we have $V = V_1 \cup V_2$ and $(\iota_1|_{U_1'}) = \varphi \cdot (\iota_2|_{U_2'})$. Hence letting $V' := \iota_1(U_1') = \iota_2(U_2')$ we have $V = \{0_1\} \,\dot\cup\, V' \,\dot\cup\, \{0_2\}$, where $V_1 = \{0_1\} \,\dot\cup\, V'$ and $V_2 = V' \,\dot\cup\, \{0_2\}$.

The topology on V is defined as follows: A subset $W \subseteq V$ is open if both $\iota_i^{-1}(W) \subseteq U_i$ are open; this is the coarsest topology on V such that both maps ι_i are continuous. Moreover, both maps ι_i are open: It suffices to consider ι_1; let $U \subseteq U_1$ be open, then we have $\iota_1^{-1}(\iota_1(U)) = U \subseteq U_1$ open, and $\iota_2^{-1}(\iota_1(U)) = \iota_2^{-1}(\iota_1(U) \cap V') = \iota_2^{-1}(\iota_1(U \cap U_1') \cap U_2') = \varphi(U \cap U_1') \subseteq U_2' \subseteq U_2$ open. Hence both maps ι_i are homeomorphisms, allowing us to identify U_i and V_i.

Next, V is connected: Let $V = W_1 \dot\cup W_2$, where $W_j \subseteq V$ are open; since $W_j \cap V_i \subseteq V_i$ is open, and V_i is irreducible, we may assume that $W_2 \cap V_1 = \emptyset$ and $W_1 \cap V_2 = \emptyset$, thus $W_1 \subseteq \{0_1\}$ and $W_2 \subseteq \{0_2\}$, a contradiction.

We proceed to define a structure sheaf $\mathcal{O}_V$ on V: Firstly, let $\mathcal{O}_{V_i}$ be the sheaf of functions on V_i obtained by pre-composition with the homeomorphism $\iota_i^{-1} \colon V_i \to U_i$. Thus $\iota_i^* \colon \mathcal{O}_{V_i} \Rightarrow \mathcal{O}_{U_i}$ is an isomorphism of sheaves, so that V_i carries the structure of an affine variety. Moreover, we get the isomorphism

$$\iota_2^* \cdot \varphi^* \cdot (\iota_1^{-1})^* \colon \mathcal{O}_{V_2}|_{V'} \Rightarrow \mathcal{O}_{U_2}|_{U_2'} = \mathcal{O}_{U_2'} \Rightarrow \mathcal{O}_{U_1'} = \mathcal{O}_{U_1}|_{U_1'} \Rightarrow \mathcal{O}_{V_1}|_{V'},$$

where on V' we indeed have $\iota_2^* \cdot \varphi^* \cdot (\iota_1^{-1})^* = (\iota_1^{-1} \cdot \varphi \cdot \iota_2)^* = (\mathrm{id}_{V'})^*$.

Now, for $W \subseteq V$ open, let $\mathcal{O}_V(W)$ be the set of all functions $f \colon W \to L$ such that both $f|_{V_i} \in \mathcal{O}_{V_i}(W \cap V_i)$. It is immediate that this defines a presheaf on V. We show that $\mathcal{O}_V$ is a sheaf:

Let $\{W_j; j \in \mathcal{J}\}$ be an open covering of W, and let $f \colon W \to L$ be a function such that $f|_{W_j} \in \mathcal{O}_V(W_j)$, for $j \in \mathcal{J}$. Then $\{W_j \cap V_i; j \in \mathcal{J}\}$ is an open covering of $W \cap V_i$, and the functions $f|_{W_j \cap V_i} = (f|_{W_j})|_{W_j \cap V_i} \in \mathcal{O}_V(W_j \cap V_i) = \mathcal{O}_{V_i}(W_j \cap V_i)$ are compatible. Thus there is $f_i \in \mathcal{O}_{V_i}(W \cap V_i)$ such that $f_i|_{W_j \cap V_i} = f|_{W_j \cap V_i}$, for $j \in \mathcal{J}$. This entails both $f|_{V_i} = f_i \in \mathcal{O}_{V_i}(W \cap V_i)$, thus $f \in \mathcal{O}_V(W)$. ♯

Hence $\{V_1, V_2\}$ is an affine open covering, where we identify $K[V_i] = K[U_i]$. Thus $(V, \mathcal{O}_V)$ is a prevariety. (Actually, in general terms, $\mathcal{O}_V$ is obtained by 'gluing' the sheaves $\mathcal{O}_{U_1}$ and $\mathcal{O}_{U_2}$ along $\varphi^* \colon \mathcal{O}_{U_2'} \Rightarrow \mathcal{O}_{U_1'}$.)

i) Let $\varphi^* \colon K[T^{\pm 1}] \to K[S^{\pm 1}] \colon T \mapsto S^{-1}$, hence $\varphi \colon U_1' \to U_2' \colon s \mapsto s^{-1}$.

Let $\mathbf{P}^1$ be the projective line, having homogeneous coordinate algebra $K[T, S]$, and let $\psi \colon \mathbf{P}^1 \to V$ be the bijection given by

$$\psi|_{D_T} \colon D_T \to V_1 \colon [t \colon s] \mapsto \iota_1(\frac{s}{t}) \quad \text{and} \quad \psi|_{D_S} \colon D_S \to V_2 \colon [t \colon s] \mapsto \iota_2(\frac{t}{s});$$

note that on $D_S \cap D_T = D_{ST}$ we have $\iota_2(\frac{t}{s}) = \iota_2(\varphi(\frac{s}{t})) = \iota_1(\frac{s}{t}) \in V'$, so that ψ is well-defined indeed. Recall that $D_T \subseteq \mathbf{P}^1$ and $D_S \subseteq \mathbf{P}^1$ are affine open such that $K[D_T] = K[S]$ and $K[D_S] = K[T]$, respectively. Thus $(\psi|_{D_T})^* = \mathrm{id}_{K[S]}$ and $(\psi|_{D_S})^* = \mathrm{id}_{K[T]}$ shows that both $\psi|_{D_T}$ and $\psi|_{D_S}$ are isomorphisms of affine varieties. Hence we conclude that ψ is an isomorphism of prevarieties, where $\mathbf{P}^1$ actually is a projective variety.

ii) Let $\varphi^* \colon K[T^{\pm 1}] \to K[S^{\pm 1}] \colon T \mapsto S$, hence $\varphi \colon U_1' \to U_2' \colon s \mapsto s$.

Writing $V' = L \setminus \{0\}$, we have $V = \{0_1\} \dot\cup (L \setminus \{0\}) \dot\cup \{0_2\}$, which is called the **affine line with one point doubled**. (This prevariety will be shown not to be an abstract variety, in particular it is not isomorphic as prevarieties to any of the varieties we have seen before.)
