Algebraic Geometry (WS 2025)

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(28.1) Limits. Let $\mathcal{I}$ be a set carrying a **partial order**, that is a reflexive, anti-symmetric and transitive relation $\leq$. Then $\mathcal{I}$ becomes a category by letting $\mathrm{Hom}_{\mathcal{I}}(i,j) = \{\rho_{ij}\}$ be a singleton set if $i \leq j$, and $\mathrm{Hom}_{\mathcal{I}}(i,j) = \emptyset$ otherwise, where concatenation is given by $\rho_{ij}\rho_{jk} = \rho_{ik}$ for $i \leq j \leq k$; in particular we have $\rho_{ii} = \mathrm{id}_i$.

Now let $\mathcal{C}$ be a category. Then an **inverse system** or **projective system** in $\mathcal{C}$ is a contravariant functor $\mathcal{I} \to \mathcal{C}$; that is it specifies a set of objects $\{A_i \in \mathcal{C}; i \in \mathcal{I}\}$, together with morphisms $\{\psi_{ji} \in \mathrm{Hom}_{\mathcal{C}}(A_j, A_i); i,j \in \mathcal{I}, j \geq i\}$, such that $\psi_{kj}\psi_{ji} = \psi_{ki}$, and $\psi_{ii} = \mathrm{id}_{A_i}$, for $k \geq j \geq i$.

Given $A \in \mathcal{C}$, let $\mathrm{Hom}_{\mathcal{C}}(A, \{A_i\})$ consist of all sets $\{\varphi_i \in \mathrm{Hom}_{\mathcal{C}}(A, A_i); i \in \mathcal{I}\}$ such that $\varphi_j\psi_{ji} = \varphi_i$, for $j \geq i$. Then the assignment

$$\mathrm{Hom}_{\mathcal{C}}(?, \{A_i\}) \colon \mathcal{C} \to \mathbf{Sets} \colon A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, \{A_i\})$$

is a contravariant functor, where a morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is mapped to the map $\varphi^* \colon \mathrm{Hom}_{\mathcal{C}}(B, \{A_i\}) \to \mathrm{Hom}_{\mathcal{C}}(A, \{A_i\})$ induced by pre-composition.

An object $P \in \mathcal{C}$, together with a **universal morphism** $\{\pi_i\} \in \mathrm{Hom}_{\mathcal{C}}(P, \{A_i\})$, is called an **(inverse) limit** or **projective limit**, if P has the following universal property: For any $A \in \mathcal{C}$ and $\{\varphi_i\} \in \mathrm{Hom}_{\mathcal{C}}(A, \{A_i\})$ there is a unique $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, P)$ such that $\varphi\pi_i = \varphi_i$, for all $i \in \mathcal{I}$.

It is immediate that the limit is unique up to unique isomorphism in $\mathcal{C}$, if it exists at all; in this case we write $P = \varprojlim\{A_i\}$. Then we have the natural bijection, given by post-composition,

$$\mathrm{Hom}_{\mathcal{C}}(A, P) \to \mathrm{Hom}_{\mathcal{C}}(A, \{A_i\}) \colon \varphi \mapsto \{\varphi\pi_i\}.$$

Recall that $\mathrm{Hom}_{\mathcal{C}}(?, P) \colon \mathcal{C} \to \mathbf{Sets} \colon A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, P)$ is a contravariant functor, where $\varphi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is mapped to $\varphi^* \colon \mathrm{Hom}_{\mathcal{C}}(B, P) \to \mathrm{Hom}_{\mathcal{C}}(A, P)$, induced by pre-composition. Then the universal property says that the universal morphism induces an equivalence of functors $\mathrm{Hom}_{\mathcal{C}}(?, P) \cong \mathrm{Hom}_{\mathcal{C}}(?, \{A_i\})$. In other words, $\mathrm{Hom}_{\mathcal{C}}(?, \{A_i\})$ is a **representable functor**, being represented by P and the universal morphism $\{\pi_i\} \in \mathrm{Hom}_{\mathcal{C}}(P, \{A_i\})$.

Example: Limits in the category of sets. i) Let $\mathcal{C} := \mathbf{Sets}$. Given sets A_i and maps $\psi_{ji} \colon A_j \to A_i$, for $j \geq i$, let $\mathcal{P} := \prod_{i \in \mathcal{I}} A_i := \{\rho \colon \mathcal{I} \to \bigcup_{i \in \mathcal{I}} A_i; \rho(i) \in A_i \text{ for } i \in \mathcal{I}\}$ be the **(Cartesian) product** of the A_i, whose elements may be written as $\mathcal{I}$-tuples, and let $\pi_i \colon \mathcal{P} \to A_i \colon x = [x_j; j \in \mathcal{I}] \mapsto x_i$ be the natural projections, for $i \in \mathcal{I}$. Then it is immediate that the subset

$$P := \{x \in \mathcal{P}; x_i = \psi_{ji}(x_j) \text{ for } j \geq i\} = \{x \in \mathcal{P}; \pi_i(x) = \psi_{ji}(\pi_j(x)) \text{ for } j \geq i\},$$

together with the induced maps $\{\pi_i\}$, is a limit in $\mathcal{C}$.

In particular, if $\mathcal{I}$ carries the **trivial** partial order, that is the only relations are $i \geq i$, for $i \in \mathcal{I}$, then the product $\mathcal{P}$ is the associated limit. If $\mathcal{I} = \emptyset$, then the (empty) product $\mathcal{P}$ boils down to a singleton set, which is a **terminal object** in $\mathcal{C}$, that is to which any set has a unique map.

ii) Let $\mathcal{I} := \{0, 1, 2\}$ be partially ordered by the non-trivial relations $1 \geq 0$ and $2 \geq 0$. Given maps $\psi_1 \colon A_1 \to A_0$ and $\psi_2 \colon A_2 \to A_0$ between sets, the associated **fibre product**, also called the **pullback** of ψ_1 and ψ_2, is defined as the associated limit in the category of sets

$$A_1 \times_{A_0} A_2 := \varprojlim\{A_0, A_1, A_2\} = \{[x_0, x_1, x_2] \in \mathcal{P}; \psi_1(x_1) = x_0 = \psi_2(x_2)\}.$$

Hence we have $A_1 \times_{A_0} A_2 \cong \{[x_1, x_2] \in A_1 \times A_2; \psi_1(x_1) = \psi_2(x_2)\}$. In particular, if A_0 is a singleton set, the product $A_1 \times A_2$ is the limit. Note that if both ψ_i are surjective, then so are the induced maps $\pi_i \colon A_1 \times_{A_0} A_2 \to A_i$.

Example: Completions of $\mathbb{Z}$. i) Let $\mathcal{I} := \mathbb{N}$, and let the partial order be given by divisibility $n \mid m$. Let $\mathbb{Z}_n := \mathbb{Z}/\langle n \rangle$, let $\psi_{mn} \colon \mathbb{Z}_m \to \mathbb{Z}_n$ be the natural ring epimorphism, for $n \mid m$, and let $\widehat{\mathbb{Z}} := \varprojlim\{\mathbb{Z}_n\} \subseteq \prod_{n \in \mathbb{N}} \mathbb{Z}_n$ be the associated limit in the category of sets.

Then $\prod_{n \in \mathbb{N}} \mathbb{Z}_n$ becomes a ring with componentwise addition and multiplication. Since the maps ψ_{mn} and π_n are ring homomorphisms, $\widehat{\mathbb{Z}}$ becomes a ring. Hence $\widehat{\mathbb{Z}}$ is a limit in the category of rings, called the **profinite completion** of $\mathbb{Z}$.

ii) Similarly, let $\mathcal{I} := \mathbb{N}_0$, and let the partial order be the natural total order. Let $p \in \mathbb{Z}$ be a prime, and considering the maps ψ_{p^m, p^n}, for $n \leq m$, let $\widehat{\mathbb{Z}}_p := \varprojlim\{\mathbb{Z}_{p^n}\} \subseteq \prod_{n \in \mathbb{N}_0} \mathbb{Z}_{p^n}$ be the associated limit in the category of sets.

Then again $\prod_{n \in \mathbb{N}_0} \mathbb{Z}_{p^n}$ becomes a ring, and hence $\widehat{\mathbb{Z}}_p$ is a limit in the category of rings, called the p**-adic completion** of $\mathbb{Z}$, or the ring of p**-adic integers**.
