Algebraic Geometry (WS 2025)

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(29.1) Products of prevarieties. We return to the earlier setting, recalling that L is assumed to be algebraically closed. We aim at showing that the category of L-prevarieties over L has (finite) products.

We would like to have $L^n \times L^m \cong L^{n+m}$ as affine L-varieties, for $n, m \in \mathbb{N}_0$, but the product topology of the Zariski topologies on L^n and L^m, resepectively, is strictly finer than the Zariski topology on L^{n+m}, whenever $n, m \geq 1$. Thus we have to define products in the categorical sense:

Lemma. Let $K = L$, let V and W be prevarieties, and let P be a product of V and W. Then P can be identified with the product $V \times W$ of sets.

Proof. For $v \in V$ there is an affine open subset $v \in U \subseteq V$, implying that $U \setminus \{v\} \subseteq U \subseteq V$ is open, thus $\{v\} \subseteq V$ is closed; note that any singleton subset of an affine L-variety is closed indeed. Hence $\{v\}$ becomes a closed subprevariety, which is affine, and thus is isomorphic to L^0. Hence there is a unique morphism $L^0 \to \{v\}$, thus we get a natural bijection $V \to \mathrm{Mor}(L^0, V)$. By the universal property of products we have bijections $P \cong \mathrm{Mor}(L^0, P) \cong \mathrm{Mor}(L^0, V) \times \mathrm{Mor}(L^0, W) \cong V \times W$, the second one being induced by post-composition with the universal morphisms $\pi_V \colon P \to V$ and $\pi_W \colon P \to W$. ♯

(29.2) Theorem: Products of affine varieties. Let $\mathbf{V} \subseteq L^n$ and $\mathbf{W} \subseteq L^m$, where $n, m \in \mathbb{N}_0$, be affine varieties having coordinate algebras $K[\mathbf{V}]$ and $K[\mathbf{W}]$.

Then $\mathbf{U} := \mathbf{V} \times \mathbf{W} \subseteq L^n \times L^m \cong L^{n+m}$ is closed, hence becomes an affine variety, which has coordinate algebra $K[\mathbf{U}] \cong K[\mathbf{V}] \otimes_K K[\mathbf{W}]$.

Proof. Let $\mathcal{X} := \{X_1, \ldots, X_n\}$ and $\mathcal{Y} := \{Y_1, \ldots, Y_m\}$ be indeterminates, let $A := K[\mathcal{X}]$ and $B := K[\mathcal{Y}]$, and let $I := \mathbf{I}_K(\mathbf{V}) \trianglelefteq A$ and $J := \mathbf{I}_K(\mathbf{W}) \trianglelefteq B$, so that $K[\mathbf{V}] = A/I$ and $K[\mathbf{W}] = B/J$.

We may identify $L^n \times L^m \cong L^{n+m}$ as sets, where L^{n+m} is an affine varietiy having coordinate algebra $C := K[\mathcal{X}, \mathcal{Y}]$. Recall that $C \cong K[\mathcal{X}] \otimes_K K[\mathcal{Y}]$, where the tensor product of K-vector spaces naturally becomes a K-algebra again; note that this coincides with the fibre sum of $K[\mathcal{X}]$ and $K[\mathcal{Y}]$ in the category of K-algebras, with respect to the structural homomorphisms $K \to K[\mathcal{X}]$ and $K \to K[\mathcal{Y}]$. Thus $f(\mathcal{X})g(\mathcal{Y}) \in C$ induces the regular function

$$(f \otimes g)^{\bullet} \colon L^n \times L^m \to L \colon [v, w] \mapsto f(v)g(w).$$

Then for $[v, w] \in L^n \times L^m$ we have $[v, w] \in \mathbf{U}$ if and only if $v \in \mathbf{V}_L(I)$ and $w \in \mathbf{V}_L(J)$. In other words, $\mathbf{U} = \mathbf{V}_L(\langle I \otimes 1, 1 \otimes J \rangle) \subseteq L^{n+m}$ is closed, such that $\langle I \otimes 1, 1 \otimes J \rangle \subseteq \mathbf{I}_K(\mathbf{U}) \trianglelefteq C$.

We have a K-bilinear map $(A/I)\times(B/J) \to C/\langle I\otimes 1, 1\otimes J\rangle \colon [f,g] \mapsto f(\mathcal{X})g(\mathcal{Y})$, which hence gives rise to an epimorphism of K-algebras

$$\alpha\colon (A/I)\otimes_K (B/J) \to C/\langle I\otimes 1, 1\otimes J\rangle \colon X_i\otimes 1 \mapsto X_i,\ 1\otimes Y_j \mapsto Y_j.$$

Concatenating α with the natural map $C/\langle I\otimes 1, 1\otimes J\rangle \to C/\mathbf{I}_K(\mathbf{U})$, we get an epimorphism $\widetilde{\alpha}\colon (A/I)\otimes_K (B/J) \to C/\mathbf{I}_K(\mathbf{U})$. We show that $\widetilde{\alpha}$ is injective:

Let $\mathcal{A} := \{f_k; k\in\mathcal{K}\} \subseteq (A/I)$ and $\mathcal{B} := \{g_l; l\in\mathcal{L}\} \subseteq (B/J)$ be K-bases, where $\mathcal{K}$ and $\mathcal{L}$ are index sets. Then $\mathcal{A}\otimes\mathcal{B} \subseteq (A/I)\otimes_K (B/J)$ is a K-basis, so that $\widetilde{\alpha}(\mathcal{A}\otimes\mathcal{B})$ spans $C/\mathbf{I}_K(\mathbf{U})$. We show that $\widetilde{\alpha}(\mathcal{A}\otimes\mathcal{B})$ is K-linearly independent:

Let $\lambda_{kl} \in K$ such that $\sum_{k\in\mathcal{K}}\sum_{l\in\mathcal{L}} \lambda_{kl} f_k(\mathcal{X}) g_l(\mathcal{Y}) = 0 \in C/\mathbf{I}_K(\mathbf{U}) = K[\mathbf{U}]$. Hence for all $v\in\mathbf{V}$ and $w\in\mathbf{W}$ we have $\sum_{k\in\mathcal{K}} f_k(v)\cdot\big(\sum_{l\in\mathcal{L}}\lambda_{kl} g_l(w)\big) = 0$. Keeping w fixed, we infer that $\sum_{k\in\mathcal{K}} f_k(\mathcal{X})\cdot\big(\sum_{l\in\mathcal{L}}\lambda_{kl} g_l(w)\big) = 0 \in K[\mathbf{V}] = A/I$. Since $\mathcal{A}\subseteq (A/I)$ is K-linearly independent, we get $\sum_{l\in\mathcal{L}}\lambda_{kl} g_l(w) = 0$, for all $k\in\mathcal{K}$. This entails $\sum_{l\in\mathcal{L}}\lambda_{kl} g_l(\mathcal{Y}) = 0 \in K[\mathbf{W}] = B/J$. Since $\mathcal{B}\subseteq (B/J)$ is K-linearly independent, we finally get $\lambda_{kl} = 0$, for all $k\in\mathcal{K}$ and $l\in\mathcal{L}$. $\sharp$

Hence both α and the natural map $C/\langle I\otimes 1, 1\otimes J\rangle \to C/\mathbf{I}_K(\mathbf{U})$ are injective, and thus are isomorphisms. Hence we have $\langle I\otimes 1, 1\otimes J\rangle = \mathbf{I}_K(\mathbf{U}) \trianglelefteq C$, and

$$K[\mathbf{U}] = C/\mathbf{I}_K(\mathbf{U}) \cong C/\langle I\otimes 1, 1\otimes J\rangle \cong (A/I)\otimes_K (B/J) = K[\mathbf{V}]\otimes_K K[\mathbf{W}].$$

$\sharp$
