Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 30** (27.01.2026)

(30.1) Theorem: Products of affine varieties. Let $\mathbf{V} \subseteq L^n$ and $\mathbf{W} \subseteq L^m$ be irreducible affine varieties having coordinate algebras $K[\mathbf{V}]$ and $K[\mathbf{W}]$, and let $\mathbf{U} := \mathbf{V} \times \mathbf{W}$ having coordinate algebra $K[\mathbf{U}] \cong K[\mathbf{V}] \otimes_K K[\mathbf{W}]$.

Then $\mathbf{U}$ is irreducible, and together with the natural projections $\pi_{\mathbf{V}}$ and $\pi_{\mathbf{W}}$ is a product of $\mathbf{V}$ and $\mathbf{W}$ in the category of prevarieties.

Proof. We keep the earlier notation used to describe the structure of the affine variety $\mathbf{U}$. Now $I \trianglelefteq A$ and $J \trianglelefteq B$ are prime. We show that $\mathbf{I}_K(\mathbf{U}) \trianglelefteq C$ is prime:

Let $h_1, h_2 \in C$ such that $h_1 h_2 \in \mathbf{I}_K(\mathbf{U})$. For $w \in \mathbf{W}$ fixed let $\mathbf{V}_i(w) := \{v \in \mathbf{V}; h_i(v,w) = 0\} = \mathbf{V}_{\mathbf{V}}(h_i(\mathcal{X}, w)) \subseteq \mathbf{V}$ closed, for $i \in \{1,2\}$. Since $h_1 h_2 \in \mathbf{I}_K(\mathbf{U})$, for any $v \in \mathbf{V}$ we have $h_1(v,w) = 0$ or $h_2(v,w) = 0$, that is $v \in \mathbf{V}_1(w)$ or $v \in \mathbf{V}_2(w)$. Thus we get $\mathbf{V} = \mathbf{V}_1(w) \cup \mathbf{V}_2(w)$. Hence, since $\mathbf{V}$ is irreducible, we have $\mathbf{V}_1(w) = \mathbf{V}$ or $\mathbf{V}_2(w) = \mathbf{V}$.

Let $\mathbf{W}_i := \{w \in \mathbf{W}; \mathbf{V}_i(w) = \mathbf{V}\} = \{w \in \mathbf{W}; h_i(v,w) = 0 \text{ for all } v \in \mathbf{V}\} = \mathbf{V}_{\mathbf{W}}(\{h_i(v, \mathcal{Y}); v \in \mathbf{V}\}) \subseteq \mathbf{W}$ closed. Then by the above we have $\mathbf{W} = \mathbf{W}_1 \cup \mathbf{W}_2$. Since $\mathbf{W}$ is irreducible, we get $\mathbf{W}_1 = \mathbf{W}$ or $\mathbf{W}_2 = \mathbf{W}$. Finally, if $\mathbf{W}_i = \mathbf{W}$ then we have $h_i(v,w) = 0$ for all $v \in \mathbf{V}$ and $w \in \mathbf{W}$, thus $h_i(v,w) \in \mathbf{I}_K(\mathbf{U})$. ♯

Next, since $\pi_{\mathbf{V}}^* \colon K[\mathbf{V}] \to K[\mathbf{U}] \colon X_i \mapsto X_i \otimes 1$ and $\pi_{\mathbf{W}}^* \colon K[\mathbf{W}] \to K[\mathbf{U}] \colon Y_j \mapsto 1 \otimes Y_j$ are homomorphisms of K-algebras, we infer that π_V and π_W are morphisms of (irreducible affine) varieties.

Now let Z be a prevariety, and let $\varphi' \colon Z \to \mathbf{V}$ and $\varphi'' \colon Z \to \mathbf{W}$ be morphisms. Since $\mathbf{U} = \mathbf{V} \times \mathbf{W}$ as sets (that is a product in the category of sets), there is a unique map $\varphi \colon Z \to \mathbf{U}$ such that $\varphi' = \varphi\pi_{\mathbf{V}}$ and $\varphi'' = \varphi\pi_{\mathbf{W}}$. Hence we have to show that φ is a morphism of prevarieties:

Since $\mathbf{U}$ is affine, it suffices to show that $\varphi^* \colon K[\mathbf{U}] = \Gamma(\mathcal{O}_{\mathbf{U}}) \to \Gamma(\mathcal{O}_Z)$ is a homomorphism of K-algebras, which just amounts to showing that it is well-defined: Recall that $K[\mathbf{U}]$ is generated by the $X_i \otimes 1$ and $1 \otimes Y_j$. Since φ' and φ'' are morphisms, we get $\varphi^*(X_i \otimes 1) = \varphi^*(\pi_{\mathbf{V}}^*(X_i)) = (\varphi\pi_{\mathbf{V}})^*(X_i) = (\varphi')^*(X_i) \in \Gamma(\mathcal{O}_Z)$ and $\varphi^*(1 \otimes Y_j) = \varphi^*(\pi_{\mathbf{W}}^*(Y_j)) = (\varphi\pi_{\mathbf{W}})^*(Y_j) = (\varphi'')^*(Y_j) \in \Gamma(\mathcal{O}_Z)$. ♯
