Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 31** (28.01.2026)

(31.1) Theorem: Products of prevarieties. Let V and W be prevarieties.
a) Then the product $P := V \times W$ as sets carries the structure of a prevariety.
b) Let $K = L$. Then P is a product of V and W in the category of prevarieties.

Proof. a) We may assume that $V \neq \emptyset \neq W$. First, we let the topology on P be defined by having the basis consisting of all subsets $V' \times W' \subseteq P$ such that both $V' \subseteq V$ and $W' \subseteq W$ are affine open, where $V' \times W'$ carries the structure of an affine variety: Noting that $(V' \times W') \cap (V'' \times W'') = (V' \cap V'') \times (W' \cap W'')$, where $V' \cap V'' \subseteq V$ and $W' \cap W'' \subseteq W$ are open, and recalling that V and W have bases consisting of affine open subsets, we infer that this indeed is a basis.

Since V is irreducible, any two non-empty open subsets of V intersect non-trivially; similarly for W. Thus any two non-empty open subsets of P belonging to the above basis intersect non-trivially, entailing that P is irreducible. Since both V and W have a finite affine open covering, this holds for P as well.

Next, as is immediately seen, the following defines a sheaf $\mathcal{O}_P$ of functions on P: For $U \subseteq P$ open let $\mathcal{O}_P(U)$ be the K-algebra of all $f\colon U \to L$ such that $f|_{U\cap(V'\times W')} \in \mathcal{O}_{V'\times W'}(U \cap (V' \times W'))$, for $V' \subseteq V$ and $W' \subseteq W$ affine open. Thus P together with $\mathcal{O}_P$ becomes a prevariety, so that $V' \times W'$ together with $\mathcal{O}_P|_{V'\times W'} = \mathcal{O}_{V'\times W'}$ is an affine open subprevariety, for all $V' \times W'$ as above.

b) Any product of V and W is necessarily isomorphic to $P = V \times W$ as sets, where π_V and π_W are the natural projections. Hence we show that P indeed is a product of V and W in the category of prevarieties:

Firstly, we show that π_V and π_W are morphisms: We consider the affine open covering by subsets $V' \times W' \subseteq P$ as above. Then $\pi_V|_{V'\times W'}\colon V' \times W' \to V'$ and $\pi_W|_{V'\times W'}\colon V' \times W' \to W'$ are morphisms, hence π_V and π_W are continuous, and by the sheaf properties are morphisms.

Now let Z be a prevariety, and let $\varphi_V : Z \to V$ and $\varphi_W : Z \to W$ be morphisms. Then there is a unique map $\varphi\colon Z \to P$ such that $\varphi_V = \varphi\pi_V$ and $\varphi_W = \varphi\pi_W$. We show that φ is a morphism of prevarieties:

We again consider the affine open covering by subsets $V' \times W' \subseteq P$ as above. Let $U := \varphi_V^{-1}(V') \cap \varphi_W^{-1}(W') \subseteq Z$ open. Then $\varphi|_U : U \to V' \times W'$ is the product map associated with the morphisms $\varphi_V|_U : U \to V'$ and $\varphi_W|_U : U \to W'$. Since $V' \times W'$ is affine, $\varphi|_U$ is a morphism. Again we infer that φ is a morphism. ♯

(31.2) Corollary. i) If $U \subseteq V$ is an open subprevariety, then so is $U \times W \subseteq V \times W$.

In particular, the product of two quasi-affine varieties is quasi-affine, and by the theorem below the product of two quasi-projective varieties is quasi-projective.

ii) If $Z \subseteq V$ is a closed subprevariety, then so is $Z \times W \subseteq V \times W$.

Proof. i) Picking affine open coverings of U and W, respectively, shows that $U \times W \subseteq P$ is open, and that $\mathcal{O}_P|_{U\times W} = \mathcal{O}_{U\times W}$.

ii) Since $(V \setminus Z) \times W \subseteq P$ is open, we conclude that $Z \times W \subseteq P$ is closed. Now P is covered by affine open subsets $V' \times W' \subseteq P$, where $V' \subseteq V$ and $W' \subseteq W$ are affine open. Then $Z' := Z \cap V' \subseteq V'$ is closed, where we may assume that $Z' \neq \emptyset$. Since $Z' \subseteq Z$ is open and Z is irreducible, Z' is irreducible as well.

Then $Z' \times W' \subseteq V' \times W'$ is closed, hence is affine with respect to $\mathcal{O}_{V'\times W'}|_{Z'\times W'}$, where global sections are given as $\Gamma(\mathcal{O}_{V'\times W'}|_{Z'\times W'}) = K[V' \times W']/\mathbf{I}_K(Z' \times W') \cong (K[V'] \otimes_K K[W'])/\langle \mathbf{I}_K(Z') \otimes 1, 1 \otimes \mathbf{I}_K(W')\rangle \cong (K[V']/\langle \mathbf{I}_K(Z')\rangle) \otimes_K K[W']$.

Similarly, $Z' \subseteq V'$ is closed, hence is affine with respect to $\mathcal{O}_{V'}|_{Z'} = \mathcal{O}_{Z'}$, where $\Gamma(\mathcal{O}_{Z'}) = K[Z'] = K[V']/\mathbf{I}_K(Z')$. Then $Z' \times W'$ is affine with respect to $\mathcal{O}_{Z'\times W'}$, global sections being given as $\Gamma(\mathcal{O}_{Z'\times W'}) = K[Z' \times W'] \cong K[Z'] \otimes_K K[W'] = (K[V']/\langle \mathbf{I}_K(Z')\rangle) \otimes_K K[W']$. Thus we infer that $\mathcal{O}_{V'\times W'}|_{Z'\times W'} = \mathcal{O}_{Z'\times W'}$.

Then from this we get $(\mathcal{O}_P|_{Z\times W})|_{Z'\times W'} = \mathcal{O}_P|_{Z'\times W'} = (\mathcal{O}_P|_{V'\times W'})|_{Z'\times W'} = \mathcal{O}_{V'\times W'}|_{Z'\times W'} = \mathcal{O}_{Z'\times W'} = \mathcal{O}_{Z\times W}|_{Z'\times W'}$. Since $Z \times W$ is covered by affine open subsets of the above form, we conclude that $\mathcal{O}_P|_{Z\times W} = \mathcal{O}_{Z\times W}$. ♯

(31.3) Theorem: Products of projective varieties. Let $\mathbf{V}$ and $\mathbf{W}$ be projective varieties. Then $\mathbf{V} \times \mathbf{W}$ is a projective variety again.

Proof. Let $\mathbf{V} \subseteq \mathbf{P}^n$ and $\mathbf{W} \subseteq \mathbf{P}^m$ be closed subprevarieties, where $n, m \in \mathbb{N}_0$. Then $\mathbf{V} \times \mathbf{W} \subseteq \mathbf{P}^n \times \mathbf{P}^m$ is a closed subprevariety as well. Since a closed subprevariety of a projective variety is projective again, it suffices to show that $\mathbf{P}^n \times \mathbf{P}^m$ is projective. To this end, we show that the latter can be identified with a closed subprevariety of $\mathbf{P}^s$, where $s := (n+1)(m+1) - 1 = n + m + nm$:

Letting $\mathcal{X}^\sharp = \{X_0, \ldots, X_n\}$, $\mathcal{Y}^\sharp = \{Y_0, \ldots, Y_m\}$, and $\mathcal{Z}^\sharp = \{Z_{00}, Z_{01}, \ldots, Z_{nm}\}$ be indeterminates associated with $\mathbf{P}^n$, $\mathbf{P}^m$, and $\mathbf{P}^s$, respectively, we consider the **Segre map** (which is immediately seen to be well-defined)

$$\sigma\colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^s\colon [[x_0\colon \ldots : x_n], [y_0\colon \ldots : y_m]] \mapsto [\ldots : x_i y_j\colon \ldots].$$

We show that σ is injective: Let $p = [x_0\colon \ldots : x_n]$, $p' = [x'_0\colon \ldots : x'_n]$, and $q = [y_0\colon \ldots : y_m]$, $q' = [y'_0\colon \ldots : y'_m]$, such that $\sigma(p, q) = \sigma(p', q')$. We may assume that $x_0 \neq 0 \neq y_0$. Then $x_0 y_0 \neq 0$ implies $x'_0 y'_0 \neq 0$, thus $x'_0 \neq 0 \neq y'_0$ as well. Hence we have $\frac{x_i y_j}{x_0 y_0} = \frac{x'_i y'_j}{x'_0 y'_0}$, entailing $\frac{x_i}{x_0} = \frac{x_i y_0}{x_0 y_0} = \frac{x'_i y'_0}{x'_0 y'_0} = \frac{x'_i}{x'_0}$ and $\frac{y_j}{y_0} = \frac{x_0 y_j}{x_0 y_0} = \frac{x'_0 y'_j}{x'_0 y'_0} = \frac{y'_j}{y'_0}$, for all i and j, thus $p = p' \in \mathbf{P}^n$ and $q = q' \in \mathbf{P}^m$. ♯

Writing $D_i := D_{X_i} \subseteq \mathbf{P}^n$, $D_j := D_{Y_j} \subseteq \mathbf{P}^m$, and $D_{ij} := D_{Z_{ij}} \subseteq \mathbf{P}^s$, we have $\sigma^{-1}(D_{ij}) = D_i \times D_j \subseteq \mathbf{P}^n \times \mathbf{P}^m$. Thus this gives rise to maps $\sigma_{ij} = \sigma|_{D_i \times D_j} \colon D_i \times D_j \to D_{ij}$, where we let $V_{ij} := \sigma_{ij}(D_i \times D_j) \subseteq D_{ij} \subseteq \mathbf{P}^s$.

For convenience letting $i = 0 = j$, and identifying $L^n \cong D_0 \subseteq \mathbf{P}^n$ and $L^m \cong D_0 \subseteq \mathbf{P}^m$ and $L^s \cong D_{00} \subseteq \mathbf{P}^s$ as affine varieties, we get the regular map $\sigma_{00} \colon L^n \times L^m \to L^s \colon [[x_1, \ldots, x_n], [y_1, \ldots, y_m]] \mapsto [z_{ij}]_{ij}$, given by $z_{ij} = x_i y_j$ and $z_{i,0} = x_i$ and $z_{0,j} = y_j$, for $i, j \geq 1$. Moreover, the comorphism $\sigma_{00}^* \colon K[\mathcal{Z}] \to K[\mathcal{X}] \otimes_K K[\mathcal{Y}]$, where $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are the respective dehomogenisations, is given by $Z_{ij} \mapsto X_i \otimes Y_j$ and $Z_{i,0} \mapsto X_i \otimes 1$ and $Z_{0,j} \mapsto 1 \otimes Y_j$, for $i, j \geq 1$.

Hence we conclude that the σ_{ij}^* are epimorphisms of K-algebras, implying that the σ_{ij} are closed embeddings of affine varieties, for all i and j, that is $V_{ij} \subseteq D_{ij}$ is closed such that $\sigma_{ij} \colon D_i \times D_j \to V_{ij}$ is an isomorphism of affine varieties.

Let $\mathbf{V} := \sigma(\mathbf{P}^n \times \mathbf{P}^m) = \bigcup_{i,j \geq 0} V_{ij} \subseteq \mathbf{P}^s$. Then we have $\mathbf{V} \cap D_{ij} = \sigma(D_i \times D_j) = V_{ij}$. Since $V_{ij} \subseteq D_{ij}$ is closed, for all i and j, we conclude that $\mathbf{V} \subseteq \mathbf{P}^s$ is closed as well, so that $\mathbf{V}$ is a projective variety. Moreover, by the sheaf properties, $\sigma \colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{V}$ is an isomorphism of prevarieties. $\sharp$
