Projective geometry

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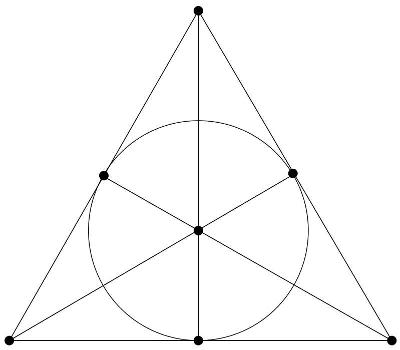
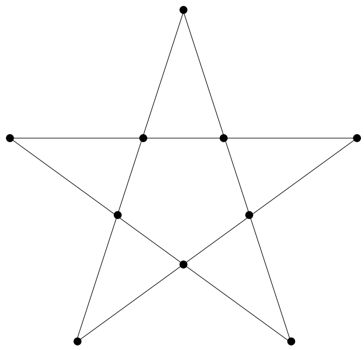




Contents

0 Introduction

Die Geometrie bedarf — ebenso wie die Arithmetik — zu ihrem folgerichtigen Aufbau nur weniger und einfacher Grundsätze. Die Grundsätze heißen Axiome der Geometrie und die Erforschung ihres Zusammenhangs ist eine Aufgabe, die seit Euklid in zahlreichen vortrefflichen Abhandlungen der mathematischen Literatur sich erörtert findet. Die bezeichnete Aufgabe läuft auf die logische Analyse unserer räumlichen Anschauung hinaus.

Die vorliegende Untersuchung ist ein neuer Versuch, für die Geometrie ein vollständiges und möglichst einfaches System von Axiomen aufzustellen und aus denselben die wichtigsten geometrischen Sätze in der Weise abzuleiten, daß dabei die Bedeutung der verschiedenen Axiomgruppen und die Tragweite der aus den einzelnen Axiomen zu ziehenden Folgerungen klar zutage tritt.

What is geometry? Geometry is the formal and logical understanding of reality ('Anschauung'). In modern terms some of its ingredients are: **objects** such as points, lines, planes, triangles, quadrangles, cubes, circles, ellipses, parabolas, hyperbolas, conics; **maps** such as symmetries, reflections, rotations, congruences, projections; in **Euclidean geometry** we for example have angles, distances, areas, volumes; and in **algebraic and differential geometry** we moreover get curves, surfaces, tangents, curvature, arc lengths.

As such, geometry has a long history, dating back to the very beginnings of mathematics. The axiomatic foundation of the geometry of the Euclidean plane was already begun in EUKLEIDES's 'Stoicheia'. But it was only in D. HILBERT's 'Grundlagen der Geometrie' that it was completed and extended to the geometry of Euclidean space; see the citation from its introduction given above. This finally paved the way for the modern axiomatic treatment of more general geometries. A few historical landmarks, with a particular view towards **projective geometry** are as follows:

• EUKLEIDES [∼300 B.C.], one of PLATON's pupils:
'Elements of Pure Mathematics' ('Stoicheia');
• APOLLONIUS, ARCHIMEDES [∼400–200 B.C.]: conics;
• PAPPUS [∼250 B.C.]: Pappus Theorem;
• [13.–14. century]: intuitive theory of perspective in painting;
• L. DA VINCI [1452–1519], A. DÜRER [1471–1528]:
mathematical theory of perspective;
• J. KEPLER [1571–1630]: conics in the theory of planetary movement;
• G. DESARGUES [1591–1661]: method of projections;
• B. PASCAL [1623–1662]: mathematical theory of conics [1639];
• A. MÖBIUS [1790–1868]: homogeneous coordinates [1827];
• F. KLEIN [1849–1925]: projective geometry over division rings [1871];
• D. HILBERT [1862–1943]: 'Grundlagen der Geometrie' [1899];
• G. FANO [1871–1952]: Fano Axiom;
• M. HALL [1910–1990]:
coordinatisation of projective planes by ternary rings [1943].

1 Planes

(1.1) Incidence geometries. a) A **geometry** $\mathcal{G}$ is a set carrying a **reflexive** and **symmetric** relation $\mathcal{I} \subseteq \mathcal{G} \times \mathcal{G}$, that is we have $[x, x] \in \mathcal{I}$ for all $x \in \mathcal{G}$, and $[y, x] \in \mathcal{I}$ whenever $[x, y] \in \mathcal{I}$. The geometry is called **finite** if $\mathcal{G}$ is a finite set.

The relation $\mathcal{I}$ is called the associated **incidence relation**; elements $x, y \in \mathcal{G}$ are called **incident** if $[x, y] \in \mathcal{I}$. A **subgeometry** $\mathcal{G}' \subseteq \mathcal{G}$ is a subset of $\mathcal{G}$ carrying the **induced** incidence relation $\mathcal{I}' := \mathcal{I} \cap (\mathcal{G}' \times \mathcal{G}')$.

If $\mathcal{G}'$ is any geometry with incidence relation $\mathcal{I}'$, then a bijective map $\alpha\colon \mathcal{G} \to \mathcal{G}'$ such that $\alpha(\mathcal{I}) = \mathcal{I}'$, where $\alpha\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}' \times \mathcal{G}'$ denotes the induced map, that is we have $[x, y] \in \mathcal{I}$ if and only if $[\alpha(x), \alpha(y)] \in \mathcal{I}'$ for all $x, y \in \mathcal{G}$, is called an **isomorphism (of geometries)**; if $\mathcal{G} = \mathcal{G}'$ then α is called an **automorphism**.

b) A **flag** is a subset $\mathcal{F} \subseteq \mathcal{G}$ such that $[x, y] \in \mathcal{I}$, for all $x, y \in \mathcal{F}$. Hence any subset of a flag is a flag again, and the set of all flags of is a partially ordered set with respect to set-theoretic inclusion. A flag $\mathcal{F}$ maximal with respect to this partial order is called a **maximal flag**, that is for all $x \in \mathcal{G} \setminus \mathcal{F}$ the set $\mathcal{F} \,\dot\cup\, \{x\} \subseteq \mathcal{G}$ is not a flag. Recall that a **partial order** $\leq$ on a set $\mathcal{M}$ is a reflexive, **anti-symmetric** and **transitive** relation on $\mathcal{M}$, that is $x \leq y$ and $y \leq x$ implies $x = y$, and $x \leq y$ and $y \leq z$ implies $x \leq z$, for all $x, y, z \in \mathcal{M}$.

c) A geometry $\mathcal{G}$ is said to have **rank** $r \in \mathbb{N}_0$ if there is a partition $\mathcal{G} = \coprod_{i=1}^{r} \mathcal{G}_i$, such that for any maximal flag $\mathcal{F} \subseteq \mathcal{G}$ we have $|\mathcal{F} \cap \mathcal{G}_i| = 1$ for all $i \in \{1, \ldots, r\}$; in particular for any maximal flag $\mathcal{F}$ we have $|\mathcal{F}| = r$. In this case distinct elements of the same type $i \in \{1, \ldots, r\}$ are not incident: Assume to the contrary that $x \neq y \in \mathcal{G}_i$ are incident, then $\{x, y\}$ is a flag, and extending it to a maximal flag $\mathcal{F}$ yields $|\mathcal{F} \cap \mathcal{G}_i| \geq 2$, a contradiction.

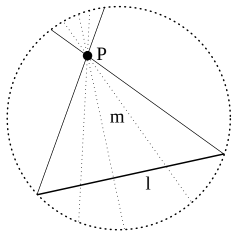
The elements of $\mathcal{G}_i$ are called the elements of $\mathcal{G}$ of **type** i. Elements of type 1, 2, 3 and r are often called **points**, **lines**, **planes** and **hyperplanes**, respectively. A geometry of rank 2 is also called an **incidence structure**, then its elements of 2 are often called **blocks**.

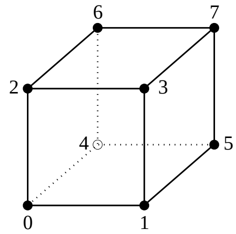
The subgeometry $\mathcal{G} \setminus \mathcal{G}_i$, for any $i \in \{1, \ldots, r\}$, is a geometry of rank $r - 1$. In particular, any geometry $\mathcal{G}$ of rank $r \geq 2$ can be considered as a geometry of rank 2, and we might be able to describe $\mathcal{G}$ in terms of its rank 2 subgeometry consisting of its points and lines, for example.

If $\mathcal{G}' = \coprod_{i=1}^{r} \mathcal{G}'_i$ is a geometry of rank r, then an isomorphism of geometries $\alpha\colon \mathcal{G} \to \mathcal{G}'$ which is **type-preserving**, that is we have $\alpha(\mathcal{G}_i) \subseteq \mathcal{G}'_i$ for all $i \in \{1, \ldots, r\}$, is called an **isomorphism (of ranked geometries)** or **collineation**; we write $\mathcal{G} \cong \mathcal{G}'$. The set $\mathrm{Aut}(\mathcal{G})$ of automorphisms of $\mathcal{G}$ is called its **automorphism group** or **collineation group**.

(1.2) Example. a) The Euclidean geometries of rank 2 and 3, called the **Euclidean plane** and the **Euclidean space**, consist of the set of all points and lines, respectively the set of all points, lines and planes, of the underlying Euclidean vector space, where the incidence relation is given by set-theoretic in-

Table 1: The hyperbolic plane and the Euclidean cube.





clusion. The elements of type 1 and 2 are the points and lines, respectively, and in the latter case the elements of type 3 are the planes.

b) Let $\mathcal{B} = \mathcal{D} \dot\cup \mathcal{L}$ be the rank 2 subgeometry of the Euclidean plane consisting of the points $\mathcal{D} := \{P \in \mathbb{R}^{2\times 1}; \|P\| < 1\}$, that is the open **unit disc**, and the lines $l \subseteq \mathbb{R}^{2\times 1}$ such that $l \cap \mathcal{D} \neq \emptyset$; see Table 1. This geometry is called the **Beltrami-Cayley-Klein** or **projective model** of the **hyperbolic plane**; it is an example of a **non-Euclidean** geometry.

Compared with the ambient Euclidean geometry, $\mathcal{B}$ has the following properties: For all points $P \neq Q \in \mathcal{D}$ there is a unique line l such that $\{P, Q\} \subseteq l$. For all lines $l \neq m \in \mathcal{L}$ we have $|l \cap m| \leq 1$; if $l \cap m = \emptyset$ then l and m are called **ultraparallel**. But, given a line $l \in \mathcal{L}$ and a point $P \in \mathcal{D}$ such that $P \notin l$, there are infinitely many lines $m \in \mathcal{L}$ such that $P \in m$ and $l \cap m = \emptyset$.

c) Let $\mathcal{C} = \mathcal{V} \dot\cup \mathcal{E} \dot\cup \mathcal{F}$ be the rank 3 subgeometry of the Euclidean space consisting of the the 8 vertices $\mathcal{V}$, the 12 edges $\mathcal{E}$ and the 6 faces $\mathcal{F}$ of the **cube**. Hence $\mathcal{C}$ can be identified with $\mathcal{V} := \{0, \ldots, 7\}$,

$$\begin{aligned} \mathcal{E} \quad := \quad & \{\{0,1\}, \{0,2\}, \{1,3\}, \{2,3\}, \{0,4\}, \{1,5\}, \\ & \{2,6\}, \{3,7\}, \{4,5\}, \{4,6\}, \{5,7\}, \{6,7\}\}, \end{aligned}$$

$$\mathcal{F} := \{\{0,1,2,3\}, \{0,1,4,5\}, \{0,2,4,6\}, \{1,3,5,7\}, \{2,3,6,7\}, \{4,5,6,7\}\},$$

where the incidence relation is given by set-theoretic inclusion; see Table 1.

(1.3) Affine Planes. An **affine plane** is a rank 2 geometry, whose elements of type 1 and 2 are called **points** and **lines**, respectively, such that the following **axioms** (A1), (A2) and (A3) are fulfilled:

(A1) Given points $P \neq Q$, there is a unique line l such that $\{P, Q\} \subseteq l$; we write $l = PQ$.

(A2: Playfair Axiom) Given a line l and a point $P \notin l$, there is a unique line $m \parallel l$ such that $P \in m$.

(A3) There are at least 3 non-collinear points.

Here, if a point P and a line l are incident, we say that P **lies on** l, and that l **contains** P; we write $P \in l$. A set of points is called **collinear** if there is a line l containing all points in this set. Lines l and m are called **parallel** if $l = m$ or their **intersection** equals $l \cap m = \emptyset$; we write $l \parallel m$.

(1.4) Properties of affine planes. a) Let $\mathcal{A}$ be an affine plane. For lines $l \not\parallel m$ we infer that $l \cap m$ is a singleton set: Assume to the contrary that there are points $P \neq Q$ such that $\{P, Q\} \subseteq l \cap m$, then (A1) implies $l = m$, a contradiction; note that here we do not need neither (A2) nor (A3).

b) Any line contains at least 2 points; thus any line is uniquely determined by its points, justifying the '$\in$' notation for incidence of points and lines:

Assume there is a line l containing no points, then by (A3) let $\{P, Q, R\}$ be non-collinear, and both $PQ \parallel l$ and $PR \parallel l$ contain P, contradicting (A2). Assume l contains precisely one point, R say, then by (A3) let $P \neq R$ and let $Q \notin PR$. By (A2) let $m \parallel QR$ contain P, hence $m \parallel l$. Since $PQ \parallel l$ also contains P, we infer $m = PQ$, which since $Q \notin m$ is a contradiction.

c) Parallelism is an equivalence relation; the **pencil of parallel lines** associated with a line l, that is the parallelism equivalence class of l, is denoted by $l_\infty := [l]_\parallel$:

Recall that an equivalence relation $\sim$ on a set $\mathcal{M}$ is a reflexive, symmetric and transitive relation on $\mathcal{M}$. To show transitivity, let l, m and n be pairwise distinct lines such that $l \parallel m$ and $m \parallel n$, and assume there is a point $P \in l \cap n$, hence $P \notin m$, contradicting (A2).

d) Any affine plane contains a **quadrangle**, that is 4 points no 3 of which are collinear; see Table 2: By (A3) there are non-collinear points $\{P, Q, R\}$. By (A2) let $l \parallel QR$ such that $P \in l$, and $m \parallel PQ$ such that $R \in m$. Assume that $l \parallel m$, then we have $QR \parallel l \parallel m \parallel PQ$, but $Q \in QR \cap PQ$, a contradiction. Hence we have $l \not\parallel m$, and thus there is a point $S \in l \cap m$. Assume that $\{P, Q, S\}$ is collinear, then $l = PQ \parallel m$, a contradiction; assume that $\{P, R, S\}$ is collinear, then $l = m$, a contradiction; assume that $\{Q, R, S\}$ is collinear, then $m = QR \parallel l$, a contradiction; in particular we have $S \notin \{P, Q, R\}$.

Indeed, the affine plane $\mathbb{A}^2(2)$ has 4 points $\{P, Q, R, S\}$ and 6 lines

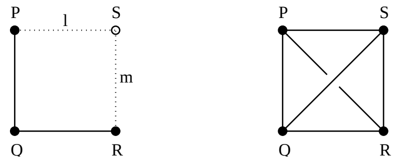
$$\{\{P, Q\}, \{Q, R\}, \{P, S\}, \{R, S\}, \{P, R\}, \{Q, S\}\},$$

each containing precisely 2 points; see Table 2. Moreover, $\mathbb{A}^2(2)$ has 3 pencils of parallel lines $(PQ)_\infty = (RS)_\infty$ and $(PS)_\infty = (QR)_\infty$ and $(PR)_\infty = (QS)_\infty$, and 12 maximal flags $\{P, PQ\}, \{P, PR\}, \{P, PS\}, \{Q, QP\}, \ldots, \{S, SR\}$. Up to isomorphism $\mathbb{A}^2(2)$ is the unique affine plane having precisely 4 points.

e) Letting $\mathcal{A}$ and $\mathcal{A}'$ be affine planes with points $\mathcal{V}$ and $\mathcal{V}'$ and lines $\mathcal{L}$ and $\mathcal{L}'$, respectively, then since any line can be identified with the points it contains, any isomorphism $\mathcal{A} \to \mathcal{A}'$ is uniquely determined by the induced bijection $\mathcal{V} \to \mathcal{V}'$.

Conversely, any bijection $\alpha \colon \mathcal{V} \to \mathcal{V}'$, such that $\{\alpha(P), \alpha(Q), \alpha(R)\} \subseteq \mathcal{V}'$ is collinear whenever $\{P, Q, R\} \subseteq \mathcal{V}$ is collinear, induces a uniquely determined

Table 2: The affine plane with 4 points.



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isomorphism $\mathcal{A} \to \mathcal{A}'$: Letting $\alpha\colon PQ \mapsto \alpha(P)\alpha(Q)$ for all $P, Q \in \mathcal{V}$ induces a well-defined map $\alpha\colon \mathcal{L} \to \mathcal{L}'$, which is surjective. Assume that $\{P, Q, R\} \subseteq \mathcal{V}$ is non-collinear such that $\alpha(PQ) = \alpha(QR)$, and by (A2) let $S \in m \in \mathcal{L}$ such that $\alpha(S) \notin \alpha(PQ) = \alpha(QR) \parallel \alpha(m)$. Then we have $PQ \parallel m \parallel QR$, a contradiction. Hence $\alpha\colon \mathcal{L} \to \mathcal{L}'$ is injective as well.

(1.5) Projective spaces. a) A **projective space** is a rank 2 geometry, whose elements of type 1 and 2 are called **points** and **lines**, respectively, such that the following axioms (P1) and (P2') are fulfilled:

(P1=A1) Given points $P \neq Q$, there is a unique line l such that $\{P, Q\} \subseteq l$.

(P2': Veblen-Young Axiom) Given pairwise distinct points $\{A, B, C, D\}$ such that $AB \not\parallel CD$, then $AC \not\parallel BD$; see Table 3.

In (P2') it suffices to assume that $A \neq B \neq D \neq C \neq A$, since if $A = D$ or $B = C$ the implication is trivial anyway. Moreover, it suffices to assume that $AB \cap CD \neq \emptyset$, since if $AB = CD$ then the implication is trivial as well. Similarly, letting $P \in AB \cap CD$, it suffices to assume that $P \notin \{A, B, C, D\}$, since otherwise the implication is trivial again. Finally, the implication is equivalent to saying $AC \cap BD \neq \emptyset$: Assume that $AC = BD$, then $\{A, B, C, D\}$ is collinear, implying $AB = CD$, a contradiction.

(P2") Given a **triangle** PAC, that is non-collinear points $\{P, A, C\}$, and a line l intersecting the **sides** PA and PC of the triangle in distinct points, that is $l \cap PA \ni B \neq D \in l \cap PC$, then $l = BD$ intersects the side AC as well.

In (P2") we may additionally assume that $\{B, D\} \cap \{P, A, C\} = \emptyset$, since otherwise the implication is trivial. Hence, since $AB = PA$ and $CD = PC$, we conclude that in the presence of (P1) the axioms (P2') and (P2") are equivalent. The idea behind (P2) is to say that distinct lines in a plane intersect without saying what a plane is.

b) A projective space is called **non-degenerate** if additionally the following axioms (P3) and (P4) are fulfilled:

(P3=A3) There are at least 3 non-collinear points.

(P4) Any line contains at least 3 points.

(P3') There are at least 2 lines.

Hence (P3) implies (P3'); conversely, if (P3') holds then (P4) implies that (P3) holds as well. Thus in the presence of (P1) and (P4) the axioms (P3) and (P3') are equivalent. Moreover, by (P4) any line is uniquely determined by its points, justifying the '$\in$' notation for incidence of points and lines.

c) A **projective plane** is a non-degenerate projective space fulfilling:

(P2) Given lines $l \neq m$, then there is a point P such that $P \in l \cap m$.

Since we may assume that $AC \neq BD$, we conclude that in the presence of (P1) the axiom (P2) implies (P2').

(1.6) Properties of projective spaces. a) Let $\mathcal{P}$ and $\mathcal{P}'$ be non-degenerate projective spaces with points $\mathcal{V}$ and $\mathcal{V}'$ and lines $\mathcal{L}$ and $\mathcal{L}'$, respectively . Then any isomorphism $\mathcal{P} \rightarrow \mathcal{P}'$ is uniquely determined by the bijection $\mathcal{V} \rightarrow \mathcal{V}'$.

Conversely, any bijection $\alpha\colon \mathcal{V} \rightarrow \mathcal{V}'$, such that $\{\alpha(P), \alpha(Q), \alpha(R)\} \subseteq \mathcal{V}'$ is collinear if and only if $\{P, Q, R\} \subseteq \mathcal{V}$ is collinear, induces a uniquely determined isomorphism $\mathcal{P} \rightarrow \mathcal{P}'$: Letting $\alpha\colon PQ \mapsto \alpha(P)\alpha(Q)$ for all $P, Q \in \mathcal{V}$ induces a well-defined map $\alpha\colon \mathcal{L} \rightarrow \mathcal{L}'$, which is surjective, and since non-collinear triples of points are mapped to non-collinear points $\alpha\colon \mathcal{L} \rightarrow \mathcal{L}'$ is injective as well.

If $\mathcal{P}$ and $\mathcal{P}'$ are projective planes, then it suffices to assume that $\alpha\colon \mathcal{V} \rightarrow \mathcal{V}'$ is a bijection such that $\{\alpha(P), \alpha(Q), \alpha(R)\} \subseteq \mathcal{V}'$ is collinear whenever $\{P, Q, R\} \subseteq \mathcal{V}$ is collinear: Assume that $\{P, Q, R\} \subseteq \mathcal{V}$ is non-collinear such that $\alpha(PQ) = \alpha(QR)$, and let $S \in \mathcal{V}$ such that $\alpha(S) \notin \alpha(PQ)$. By (P2) let $T \in \mathcal{V}$ such that $T \in PS \cap QR$, hence $P \neq T$ and $\{\alpha(P), \alpha(T)\} \subseteq \alpha(PS) \cap \alpha(PQ) = \alpha(PS) \cap \alpha(QR)$, a contradiction. Hence $\alpha\colon \mathcal{L} \rightarrow \mathcal{L}'$ is injective as well.

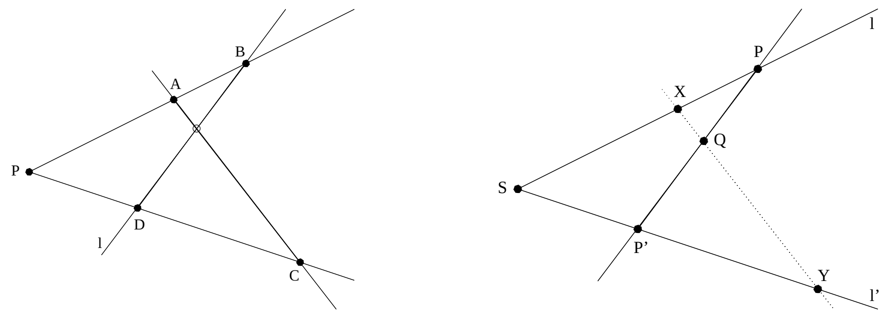
b) Let $\mathcal{P}$ be a non-degenerate projective space with points $\mathcal{V}$, and let l and l' be lines in $\mathcal{P}$. Then there is a bijection $\pi\colon \{X \in \mathcal{V}; X \in l\} \rightarrow \{Y \in \mathcal{V}; Y \in l'\}$:

We may assume that $l \neq l'$. Let firstly $S \in l \cap l'$; see Table 3. Then let $S \neq P \in l$ and $S \neq P' \in l'$, hence $\{S, P, P'\}$ is non-collinear. Let $Q \in PP'$ such that $P \neq Q \neq P'$, hence we have $Q \notin l$ and $Q \notin l'$. Hence for any $X \in l$ the line XQ intersects l and PP' in distinct points, thus by (P2") there is a unique point $X' \in XQ \cap l'$, defining a map $\pi\colon X \rightarrow X'$. Similarly, for any $Y \in l'$ the line YQ intersects l' and PP' in distinct points, thus there is a unique point $Y' \in YQ \cap l'$, defining a map $\pi'\colon Y \rightarrow Y'$. Since π and π' are mutually inverse to each other, π is as desired.

Secondly let $l \parallel l'$. Then let $P \in l$ and $P' \in l'$. Then we have $l \neq PP' \neq l'$, and by the above there are bijections $\pi\colon \{X \in \mathcal{V}; X \in l\} \rightarrow \{Z \in \mathcal{V}; Z \in PP'\}$ and $\pi'\colon \{Y \in \mathcal{V}; Y \in l'\} \rightarrow \{Z \in \mathcal{V}; Z \in PP'\}$, hence $\pi'^{-1}\pi$ is as desired.

(1.7) Duality. a) Let $\mathcal{G}$ be a geometry of rank 2. Then the **dual** geometry $\mathcal{G}^*$ has the lines and points of $\mathcal{G}$ as points and lines, and a point and a line are incident in $\mathcal{G}^*$ if and only if they are incident in $\mathcal{G}$. Hence we have $(\mathcal{G}^*)^* = \mathcal{G}$.

Table 3: The Veblen-Young Axiom.



Thus any **statement on geometries** holds for $\mathcal{G}$ if and only if the **dual** statement, obtained by interchanging the roles of 'points' and 'lines', holds for $\mathcal{G}^*$.

b) Any projective plane $\mathcal{P}$ fulfils the following axioms dual to (P1–4):

(P1*) Given lines $l \neq m$, then there is a unique point P such that $P \in l \cap m$.

(P2*) Given points $P \neq Q$, then there is a line l such that $\{P, Q\} \subseteq l$.

(P3*) There are at least 3 **non-concurrent** lines, that is having no point in common.

(P4*) Any point lies on at least 3 lines.

Axioms (P1) and (P2) imply (P1*), and (P1) implies (P2*). By (P3) there are 3 non-collinear points, hence by (P1) these define 3 lines having no common point, thus (P3*) holds. To verify (P4*), let P be a point, then by (P3*) there is a line l not containing P, and since by (P4) l contains at least 3 points, by (P1) there are at least 3 lines containing P. $\sharp$

Note that in general $\mathcal{P} \not\cong \mathcal{P}^*$. Still, we have the following **principle of duality:** A statement on geometries can be deduced from the axioms (P1–4) if and only if the dual statement can.

2 Completion

(2.1) From affine to projective planes. Any affine plane $\mathcal{A}$, with points $\mathcal{V}$ and lines $\mathcal{L}$, can be completed to yield a projective plane as follows:

Recall that any line $l \in \mathcal{L}$ can be identified with the points it contains, hence can be considered as $l \subseteq \mathcal{V}$. The pencil of parallel lines $l_\infty \subseteq \mathcal{L}$ is called the associated **ideal point** or **point at infinity** in the **direction** of l; let $\mathcal{V}_\infty := \{l_\infty \subseteq \mathcal{L}; l \in \mathcal{L}\}$ be the set of all ideal points.

Let $\widehat{\mathcal{A}}$ be the following rank 2 geometry: The points of $\widehat{\mathcal{A}}$ are given as $\mathcal{V} \,\dot\cup\, \mathcal{V}_\infty$, that is the **finite** and ideal points of $\mathcal{A}$. The lines of $\widehat{\mathcal{A}}$ are given as the **finite** lines $\widehat{l} := l \,\dot\cup\, \{l_\infty\} \subseteq \mathcal{V} \,\dot\cup\, \mathcal{V}_\infty$ for $l \in \mathcal{L}$, that is the affine lines together with their associated ideal points, and the **line at infinity** $\mathcal{V}_\infty$; hence any finite line $\widehat{l}$ contains a single ideal point, and $\mathcal{V}_\infty$ is the unique line without finite points. The incidence relation given by set-theoretic inclusion, $\widehat{\mathcal{A}}$ is a projective plane:

i) Let $P \neq Q$ be finite points, and let $l_\infty \neq m_\infty$ be ideal points. Then $\widehat{PQ}$ is the unique line containing $\{P, Q\}$, and $\mathcal{V}_\infty$ is the unique line containing $\{l_\infty, m_\infty\}$. There is a finite line $k \parallel l$ such that $P \in k$, hence from $k_\infty = l_\infty$ we get $\{P, l_\infty\} \subseteq \widehat{k} = k \,\dot\cup\, \{k_\infty\}$; if conversely $\widehat{k}$ contains $\{P, l_\infty\}$, then $P \in k$ and $k_\infty = l_\infty$, that is $k \parallel l$, implying uniqueness of k. This shows (P1).

ii) Let $\widehat{l} \neq \widehat{m}$ be finite lines. If $l \nparallel m$ then there is a finite point $P \in \widehat{l} \cap \widehat{m}$, if $l \parallel m$ then $l_\infty = m_\infty \in \widehat{l} \cap \widehat{m}$, and we have $l_\infty \in \widehat{l} \cap \mathcal{V}_\infty$. This shows (P2).

iii) Let $\{P, Q, R\}$ be finite points, non-collinear in $\mathcal{A}$. Since $\mathcal{V}_\infty$ is the only line whose finite points do not form a finite line, we conclude that $\{P, Q, R\}$ is non-collinear in $\widehat{\mathcal{A}}$ as well. This shows (P3).

iv) Let $\widehat{l}$ be a finite line. Since l contains at least 2 points, $\widehat{l} = l \,\dot\cup\, \{l_\infty\}$ contains at least 3 points. Moreover, letting $\{P, Q, R\}$ be non-collinear finite points, the affine lines PQ, PR and QR are pairwise non-parallel, hence there are at least 3 pencils of parallel lines. This shows (P4). $\sharp$

In particular, if $\mathcal{A}'$ is an affine plane, then any isomorphism $\alpha\colon \mathcal{A} \to \mathcal{A}'$ extends uniquely to an isomorphism $\widehat{\alpha}\colon \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}'$: Since the lines of $\widehat{\mathcal{A}}$ are given as $\widehat{l} = l \,\dot\cup\, \{l_\infty\}$ for $l \in \mathcal{L}$, we necessarily have $\widehat{\alpha}(\widehat{l}) = \widehat{\alpha(l)}$ and $\widehat{\alpha}(l_\infty) = \alpha(l)_\infty$. Conversely, for lines $l, m \in \mathcal{L}$ we have $l \parallel m$ in $\mathcal{A}$ if and only if $\alpha(l) \parallel \alpha(m)$ in $\mathcal{A}'$, hence α induces a bijection $\mathcal{V}_\infty \to \mathcal{V}'_\infty\colon l_\infty \mapsto \alpha(l)_\infty$, thus bijections between the points as well as between the lines of $\widehat{\mathcal{A}}$ and of $\widehat{\mathcal{A}}'$, transporting incidence.

(2.2) From projective to affine planes. Given a projective plane $\mathcal{P}$, with points $\mathcal{V}$ and lines $\mathcal{L}$, let $l_0 \in \mathcal{L}$ be fixed, and let $\mathcal{P}_0$ be the following rank 2 geometry: The points of $\mathcal{P}_0$ are given as $\{P \in \mathcal{V}; P \notin l_0\}$. The lines are given as $l^\bullet := l \setminus (l \cap l_0)$ for $l_0 \neq l \in \mathcal{L}$; hence any $l^\bullet$ contains at least 2 points, thus can be identified with the points it contains, and for $l_0 \neq l' \in \mathcal{L}$ we have $l^\bullet = (l')^\bullet$ if and only if $l = l'$. The incidence relation given by set-theoretic inclusion, $\mathcal{P}_0$ is an affine plane:

i) Let $P \neq Q$ be points, not lying on l_0. Then there is a unique line $l \in \mathcal{L}$ containing $\{P, Q\}$, hence $l \neq l_0$ and thus $\{P, Q\} \subseteq l^\bullet$. This shows (A1).

ii) Let $l_0 \neq l \in \mathcal{L}$. Since distinct lines in $\mathcal{L}$ intersect, for $l \neq l' \in \mathcal{L} \setminus \{l_0\}$ we have $l^\bullet \parallel (l')^\bullet$ if and only if $l \cap l' \subseteq l_0$. Hence, if P is any point such that $P \notin l_0$ and $P \notin l$, then letting $Q \in l \cap l_0$ we infer that $(PQ)^\bullet = (PQ) \setminus \{Q\}$ is the unique line parallel to $l^\bullet = l \setminus \{Q\}$ containing P. This shows (A2).

iii) Since there are at least 2 lines, let $l_0 \neq l \in \mathcal{L}$ and $P \in l \cap l_0$. Since any line in $\mathcal{L}$ contains at least 3 points, let $\{P, P', P''\} \subseteq l_0$ be pairwise distinct, and let

$Q \in l^\bullet = l \setminus \{P\}$. Let $l' := QP' \in \mathcal{L}$ and $l'' := QP'' \in \mathcal{L}$, as well as $Q \neq Q' \in (l')^\bullet = l' \setminus \{P'\}$ and $Q \neq Q'' \in (l'')^\bullet = l'' \setminus \{P''\}$, hence $QQ' = l' \neq l'' = QQ''$ shows that $\{Q, Q', Q''\}$ is non-collinear. This shows (A3). ♯

In particular, any automorphism $\alpha \colon \mathcal{P} \to \mathcal{P}$ such that $\alpha(l_0) = l_0$ restricts to an automorphism $\alpha_0 \colon \mathcal{P}_0 \to \mathcal{P}_0$: Since α_0 permutes the points lying on l_0, it restricts to bijections on $\{P \in \mathcal{V}; P \notin l_0\}$ and on $\{l^\bullet; l_0 \neq l \in \mathcal{L}\}$, where incidence is transported.

(2.3) Back and forth. a) Given an affine plane $\mathcal{A}$, then the affine plane $(\widehat{\mathcal{A}})_\infty$, obtained by completion first, followed by deletion of the line at infinity $\mathcal{V}_\infty$ from the projective plane $\widehat{\mathcal{A}}$, can be identified with $\mathcal{A}$ again; note that in general $(\widehat{\mathcal{A}})_0$ obtained by deleting an arbitrary line l_0 from $\widehat{\mathcal{A}}$ cannot be identified with $\mathcal{A}$:

Since $\mathcal{V}_\infty$ is the set of ideal points of $\mathcal{A}$, the points of $(\widehat{\mathcal{A}})_\infty$ and $\mathcal{A}$ coincide. Moreover, since we are deleting the line $\mathcal{V}_\infty$, the lines of $(\widehat{\mathcal{A}})_\infty$ are obtained from the lines l of $\mathcal{A}$ as $(\widehat{l})^\bullet = (l \,\dot\cup\, \{l_\infty\})^\bullet = (l \,\dot\cup\, \{l_\infty\}) \setminus \{l_\infty\} = l$. Finally, for both $\mathcal{A}$ and $(\widehat{\mathcal{A}})_\infty$ the incidence relation is given by set-theoretic inclusion.

b) Conversely, given a projective plane $\mathcal{P}$ and a line $l_0 \in \mathcal{L}$, the projective plane $\widehat{\mathcal{P}_0}$, obtained by deleting l_0 first and subsequently completing the affine plane $\mathcal{P}_0$, can be identified with $\mathcal{P}$ again:

For $l_0 \neq l \in \mathcal{L}$ let $P_l \in l \cap l_0$, yielding $\pi \colon \mathcal{L} \setminus \{l_0\} \to \{P \in \mathcal{V}; P \in l_0\} \colon l \mapsto P_l$. Since there is a point not lying on l_0, the map π is surjective. Hence for $l \neq l' \in \mathcal{L} \setminus \{l_0\}$ we have $l^\bullet \parallel (l')^\bullet$ if and only if $l \cap l' \subseteq l_0$, which holds if and only if $P_l \in l \cap l_0 = l' \cap l_0 \ni P_{l'}$, that is if and only if $\pi(l) = \pi(l')$. Hence π induces a bijection $\{(l^\bullet)_\infty; l \in \mathcal{L} \setminus \{l_0\}\} \to \{P \in \mathcal{V}; P \in l_0\} \colon (l^\bullet)_\infty \mapsto \pi(l) = P_l$ from the set of ideal points of $\mathcal{P}_0$, that is the pencils of lines, to the points lying on l_0.

Thus the line at infinity of $\widehat{\mathcal{P}_0}$ can be identified via π with l_0. Using this identification, the points of $\widehat{\mathcal{P}_0}$ are $\{P \in \mathcal{V}; P \notin l_0\} \,\dot\cup\, \{\pi(l) \in \mathcal{V}; l \in \mathcal{L} \setminus \{l_0\}\} = \{P \in \mathcal{V}; P \notin l_0\} \,\dot\cup\, \{P \in \mathcal{V}; P \in l_0\} = \mathcal{V}$, and the finite lines of $\widehat{\mathcal{P}_0}$ are obtained from the lines $l_0 \neq l$ of $\mathcal{P}$ as $l^\bullet \,\dot\cup\, \{\pi(l)\} = (l \setminus \{\pi(l)\}) \,\dot\cup\, \{\pi(l)\} = l$. Finally, for both $\mathcal{P}$ and $\widehat{\mathcal{P}_0}$ the incidence relation is given by set-theoretic inclusion.

c) Hence any projective plane can be considered as a completion of an affine plane, and since any affine plane has at least 4 points, and any line of a projective plane contains at least 3 points, any projective plane has at least 7 points.

For example, completion of the affine plane $\mathbb{A}^2(2)$ yields the projective plane $\mathbb{P}^2(2) := \widehat{\mathbb{A}^2(2)}$, also called the **Fano plane**; it has 7 points, each lying on 3 lines, and 7 lines, each containing 3 points; see Table 4.

Conversely, if $\mathcal{P}$ is a projective plane with precisely 7 points, then deleting any line l_0 we obtain an affine plane $\mathcal{P}_0$. Since l_0 contains at least 3 points, and $\mathcal{P}_0$ has at least 4 points, we infer that $\mathcal{P}_0$ has precisely 4 points, thus we have $\mathcal{P}_0 \cong \mathbb{A}^2(2)$. Hence we conclude that $\mathcal{P} \cong \widehat{\mathcal{P}_0} \cong \widehat{\mathbb{A}^2(2)} = \mathbb{P}^2(2)$ is uniquely determined up to isomorphism.

(2.4) Planes over fields. Let $\mathbb{F}$ be a field; this in particular encompasses the motivating case of the real number field $\mathbb{R}$.

a) The **affine plane** $\mathbb{A}^2(\mathbb{F})$ **over** $\mathbb{F}$, also called the **affine space of dimension** 2 **over** $\mathbb{F}$, is given as follows: Its set of points being the 2-dimensional $\mathbb{F}$-vector space $\mathbb{F}^{2\times 1}$, its lines are given as $\mathcal{L}(a,b;c) := \{[x,y]^{\mathrm{tr}} \in \mathbb{F}^{2\times 1}; ax+by+c=0\} = \{[x,y]^{\mathrm{tr}} \in \mathbb{F}^{2\times 1}; [a,b]\cdot[x,y]^{\mathrm{tr}}+c=0\} = \{[x,y]^{\mathrm{tr}} \in \mathbb{F}^{2\times 1}; [a,b,c]\cdot[x,y,1]^{\mathrm{tr}}=0\}$ where $0 \neq [a,b] \in \mathbb{F}^2$ and $c \in \mathbb{F}$. Letting $A := \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \in \mathbb{F}^{2\times 2}$, where hence $\mathrm{rk}(A) \geq 1$, we have $\mathcal{L}(a,b;c) = \mathcal{L}(a',b';c')$ if and only if the system of linear equations $A\cdot[x,y]^{\mathrm{tr}} + [c,c']^{\mathrm{tr}} = 0$ has more than one solution, which holds if and only if the extended matrix $\left[\begin{array}{c|c} A & \begin{matrix} c \\ c' \end{matrix} \end{array}\right] \in \mathbb{F}^{2\times 3}$ has rank 1, which in turn holds if and only if $\langle [a,b,c]\rangle_{\mathbb{F}} = \langle [a',b',c']\rangle_{\mathbb{F}} \leq \mathbb{F}^3$; in this case we have $\mathrm{rk}(A) = 1$.

Lines $\mathcal{L}(a,b;c) \neq \mathcal{L}(a',b';c')$, that is the extended matrix has rank 2, intersect if and only if the system of linear equations $A\cdot[x,y]^{\mathrm{tr}} + [c,c']^{\mathrm{tr}} = 0$ has a solution, which holds if and only if $\mathrm{rk}(A) = 2$; in this case the solution is unique. Hence we have $\mathcal{L}(a,b;c) \parallel \mathcal{L}(a',b';c')$ if and only if $\langle [a,b]\rangle_{\mathbb{F}} = \langle [a',b']\rangle_{\mathbb{F}} \leq \mathbb{F}^2$. The incidence relation being set-theoretic inclusion, $\mathbb{A}^2(\mathbb{F})$ indeed is an affine plane:

i) Let $[x,y]^{\mathrm{tr}} \neq [x',y']^{\mathrm{tr}} \in \mathbb{F}^{2\times 1}$. Then the matrix $B := \begin{bmatrix} x & x' \\ y & y' \\ 1 & 1 \end{bmatrix} \in \mathbb{F}^{3\times 2}$ has rank 2, and thus the system of linear equations $[a,b,c]\cdot B = 0$ has a 1-dimensional $\mathbb{F}$-subspace of solutions, different from $\langle [0,0,1]\rangle_{\mathbb{F}} \leq \mathbb{F}^3$. Hence there is a unique line containing $\{[x,y]^{\mathrm{tr}}, [x',y']^{\mathrm{tr}}\}$, showing (A1).

ii) Let $\mathcal{L}(a,b;c)$ be a line and let $[x,y]^{\mathrm{tr}} \in \mathbb{F}^{2\times 1}$ be a point. The lines parallel to $\mathcal{L}(a,b;c)$ being given as $\mathcal{L}(a,b;c')$ for $c' \in \mathbb{F}$, we conclude that $\mathcal{L}(a,b;-[a,b]\cdot[x,y]^{\mathrm{tr}})$ is the unique line parallel to $\mathcal{L}(a,b;c)$ containing $[x,y]^{\mathrm{tr}}$, showing (A2).

iii) The points $\{[0,0]^{\mathrm{tr}}, [1,0]^{\mathrm{tr}}, [0,1]^{\mathrm{tr}}\} \subseteq \mathbb{F}^{2\times 1}$ are non-collinear, the former two being contained in $\mathcal{L}(0,1;0)$ but not the latter, showing (A3).

b) The completion $\widehat{\mathbb{A}^2(\mathbb{F})}$ of $\mathbb{A}^2(\mathbb{F})$ is called the **projective plane over** $\mathbb{F}$ or the **projective space of dimension** 2 **over** $\mathbb{F}$. The pencils of parallel lines are in bijection with the 1-dimensional $\mathbb{F}$-subspaces $\langle [a,b]\rangle_{\mathbb{F}} \leq \mathbb{F}^2$; the number $-\frac{a}{b} \in \mathbb{F} \,\dot\cup\, \{\infty\}$ describes the **slope** of the lines in question. Hence the points of $\widehat{\mathbb{A}^2(\mathbb{F})}$ are given as $\mathbb{F}^{2\times 1} \,\dot\cup\, \mathcal{V}_\infty$, where $\mathcal{V}_\infty := \{\langle [a,b]\rangle_{\mathbb{F}}; [a,b] \neq 0\}$ is the line at infinity, and the finite lines are given as $\widehat{\mathcal{L}}(a,b;c) := \mathcal{L}(a,b;c) \,\dot\cup\, \{\langle [a,b]\rangle_{\mathbb{F}}\}$.

(2.5) The homogeneous model. Let $\mathbb{A}^3(\mathbb{F})$ be the **affine space of dimension** 3 over $\mathbb{F}$. Its set of points is the 3-dimensional $\mathbb{F}$-vector space $\mathbb{F}^{3\times 1}$; hence in the case $\mathbb{F} = \mathbb{R}$ we just consider Euclidean space.

Let $\mathbb{P}^2(\mathbb{F})$ be the rank 2 subgeometry of $\mathbb{A}^3(\mathbb{F})$, whose points and lines are the affine lines and planes containing 0, respectively. Thus the points are the 1-dimensional $\mathbb{F}$-subspaces $\langle [x,y,z]^{\mathrm{tr}}\rangle_{\mathbb{F}} \leq \mathbb{F}^{3\times 1}$ where $0 \neq [x,y,z]^{\mathrm{tr}} \in$

$\mathbb{F}^{3\times 1}$, and the lines are the 2-dimensional $\mathbb{F}$-subspaces $\mathcal{H}(a,b,c) := \{[x,y,z]^{\text{tr}} \in \mathbb{F}^{3\times 1}; [a,b,c]\cdot[x,y,z]^{\text{tr}} = 0\} \leq \mathbb{F}^{3\times 1}$ where $0 \neq [a,b,c] \in \mathbb{F}^3$, the incidence relation being given by set-theoretic inclusion. We have $\mathcal{H}(a,b,c) = \mathcal{H}(a',b',c')$ if and only if $\begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} \in \mathbb{F}^{2\times 3}$ has rank 1, which holds if and only if $\langle [a,b,c]\rangle_{\mathbb{F}} = \langle [a',b',c']\rangle_{\mathbb{F}} \leq \mathbb{F}^3$. Intersecting with the plane $\mathcal{A} := \{[x,y,1]^{\text{tr}} \in \mathbb{F}^{3\times 1}; x,y \in \mathbb{F}\}$ induces an isomorphism $\alpha\colon \mathbb{P}^2(\mathbb{F}) \to \widehat{\mathbb{A}^2(\mathbb{F})}$:

i) Let $\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}}$ be a point of $\mathbb{P}^2(\mathbb{F})$. If $z \neq 0$ then we have $\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}} \cap \mathcal{A} = \{[\frac{x}{z}, \frac{y}{z}, 1]^{\text{tr}}\}$, and we let $\alpha(\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}}) := [\frac{x}{z}, \frac{y}{z}]^{\text{tr}} \in \mathbb{F}^{2\times 1}$. If $z = 0$ then we have $\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}} \cap \mathcal{A} = \emptyset$, and since $[x,y]^{\text{tr}} \neq 0$ letting $\langle [a,b]\rangle_{\mathbb{F}} \leq \mathbb{F}^2$ be the 1-dimensional $\mathbb{F}$-subspace of solutions of the linear equation $[a,b]\cdot[x,y]^{\text{tr}} = 0$, we let $\alpha(\langle [x,y,0]^{\text{tr}}\rangle_{\mathbb{F}}) := \langle [a,b]\rangle_{\mathbb{F}}$, an ideal point of $\mathbb{A}^2(\mathbb{F})$. Hence α is a bijection between the points of $\mathbb{P}^2(\mathbb{F})$ and of $\widehat{\mathbb{A}^2(\mathbb{F})}$:

ii) Let $\mathcal{H}(a,b,c)$ be a line of $\mathbb{P}^2(\mathbb{F})$. Then we have $\mathcal{H}(a,b,c) \cap \mathcal{A} = \emptyset$ if and only if $\langle [a,b,c]\rangle_{\mathbb{F}} = \langle [0,0,1]\rangle_{\mathbb{F}}$. Hence if $\mathcal{H}(a,b,c) \neq \mathcal{H}(0,0,1)$ we let $\alpha(\mathcal{H}(a,b,c)) := \{[x,y]^{\text{tr}} \in \mathbb{F}^{2\times 1}; [a,b,c]\cdot[x,y,1]^{\text{tr}} = 0\} \,\dot\cup\, \{\langle [a,b]\rangle_{\mathbb{F}}\} = \widehat{\mathcal{L}}(a,b;c)$, a finite line in $\widehat{\mathbb{A}^2(\mathbb{F})}$, and we let $\alpha(\mathcal{H}(0,0,1)) := \mathcal{V}_\infty$, the line at infinity of $\widehat{\mathbb{A}^2(\mathbb{F})}$. Since $\mathcal{H}(a,b,c) = \mathcal{H}(a',b',c')$ if and only if $\langle [a,b,c]\rangle_{\mathbb{F}} = \langle [a',b',c']\rangle_{\mathbb{F}}$, and $\mathcal{L}(a,b;c) = \mathcal{L}(a',b';c')$, for $[a,b] \neq 0$, if and only if $\langle [a,b,c]\rangle_{\mathbb{F}} = \langle [a',b',c']\rangle_{\mathbb{F}}$, we infer that α is a bijection between the lines of $\mathbb{P}^2(\mathbb{F})$ and of $\widehat{\mathbb{A}^2(\mathbb{F})}$.

iii) A point $\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}}$ lies on a line $\mathcal{H}(a,b,c)$ if and only if $[a,b,c]\cdot[x,y,z]^{\text{tr}} = 0$. Hence, if $[a,b] \neq [0,0]$ and $z \neq 0$ this is equivalent to $[a,b]\cdot[\frac{x}{z}, \frac{y}{z}]^{\text{tr}} + c = 0$, that is $\alpha(\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}})$ lying on $\alpha(\mathcal{H}(a,b,c)) = \widehat{\mathcal{L}}(a,b;c)$, while if $z = 0$ this is equivalent to $[a,b]\cdot[x,y]^{\text{tr}} = 0$, that is $\alpha(\langle [x,y,0]^{\text{tr}}\rangle_{\mathbb{F}}) = \langle [a,b]\rangle_{\mathbb{F}}$ lying on $\alpha(\mathcal{H}(a,b,c)) = \widehat{\mathcal{L}}(a,b;c) = \mathcal{L}(a,b;c) \,\dot\cup\, \{\langle [a,b]\rangle_{\mathbb{F}}\}$; if $[a,b] = [0,0]$ this is equivalent to $z = 0$, which in turn is equivalent to $\alpha(\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}})$ being an ideal point of $\mathbb{A}^2(\mathbb{F})$, that is $\alpha(\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}})$ lying on $\alpha(\mathcal{H}(0,0,1)) = \mathcal{V}_\infty$. ♯

This shows that $\mathbb{P}^2(\mathbb{F})$ is a projective plane and can be identified with $\widehat{\mathbb{A}^2(\mathbb{F})}$, being called its **homogeneous model**. The tuples $[x,y,z]^{\text{tr}} \in \mathbb{F}^{3\times 1}$ are called **homogeneous coordinates** of $\alpha(\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}}) \in \widehat{\mathbb{A}^2(\mathbb{F})}$; note that the latter are only unique up to non-zero scalars. Since $[a,b,c]\cdot[x,y,z]^{\text{tr}} = 0$ if and only if $[x,y,z]\cdot[a,b,c]^{\text{tr}} = 0$, mapping $\langle [x,y,z]^{\text{tr}}\rangle_{\mathbb{F}} \mapsto \mathcal{H}(x,y,z)$ and $\mathcal{H}(a,b,c) \mapsto \langle [a,b,c]^{\text{tr}}\rangle_{\mathbb{F}}$ defines a **polarity**, that is an isomorphism $\mathbb{P}^2(\mathbb{F}) \to \mathbb{P}^2(\mathbb{F})^*$.

(2.6) Example: The real projective plane. Considering the **unit sphere** $\mathbb{S}^2(\mathbb{R}) := \{[x,y,z]^{\text{tr}} \in \mathbb{R}^{3\times 1}; x^2+y^2+z^2 = 1\}$ in Euclidean space yields the following geometry $\mathcal{S}$ of rank 2: The points of $\mathcal{S}$ are the pairs of **antipodal** points $\pm[x,y,z]^{\text{tr}} \in \mathbb{S}^2(\mathbb{R})$, that is the intersections of the points of $\mathbb{P}^2(\mathbb{R})$ with $\mathbb{S}^2(\mathbb{R})$. The lines of $\mathcal{S}$ are the **great circles** on $\mathbb{S}^2(\mathbb{R})$, that is the intersections of the lines of $\mathbb{P}^2(\mathbb{R})$ with $\mathbb{S}^2(\mathbb{R})$. This defines an isomorphism $\mathbb{P}^2(\mathbb{R}) \to \mathcal{S}$, being called the **sphere model** of the real projective plane.

(2.7) Example: Planes over finite fields. a) Let q be a prime power, and let $\mathbb{F}_q$ be the finite field with q elements. Then the affine plane $\mathbb{A}^2(q) := \mathbb{A}^2(\mathbb{F}_q)$ has q^2 points and $\frac{q(q^2-1)}{q-1} = q^2+q$ lines, each line containing q points, and each point lying on $\frac{q^2-1}{q-1} = q+1$ lines. Moreover, there are $\frac{q^2-1}{q-1} = q+1$ pencils of parallel lines, each of which consists of q lines.

The affine space $\mathbb{A}^3(\mathbb{F}_q)$ has $\frac{q^3-1}{q-1} = q^2+q+1 = \frac{(q^3-1)(q^3-q)}{(q^2-1)(q^2-q)}$ 1-dimensional and 2-dimensional $\mathbb{F}_q$-subspaces each, where any 2-dimensional $\mathbb{F}_q$-subspace or $\mathbb{F}_q$-quotient space contains precisely $\frac{q^2-1}{q-1} = q+1$ many 1-dimensional $\mathbb{F}_q$-subspaces. Hence the projective plane $\mathbb{P}^2(q) := \mathbb{P}^2(\mathbb{F}_q)$ has q^2+q+1 points and lines each, any line containing $q+1$ points, and any point lying on $q+1$ lines.

b) In particular, for the field $\mathbb{F}_2 = \{0,1\}$ with 2 elements, the affine plane $\mathbb{A}^2(2)$ has 4 points $\mathbb{F}_2^{2\times 1} = \{[0,0]^{\text{tr}}, [1,0]^{\text{tr}}, [0,1]^{\text{tr}}, [1,1]^{\text{tr}}\}$ and 6 lines

$$\mathcal{L}(1,0;0),\quad \mathcal{L}(1,0;1);\quad \mathcal{L}(0,1;0),\quad \mathcal{L}(0,1;1);\quad \mathcal{L}(1,1;0),\quad \mathcal{L}(1,1;1),$$

where the 3 pencils of parallel lines are also indicated. Thus we indeed recover the affine plane with 4 points and its completion encountered earlier.

The points of $\mathbb{P}^2(2)$ are the 1-dimensional $\mathbb{F}_2$-subspaces of $\mathbb{F}_2^{3\times 1}$. In homogeneous coordinates the finite and ideal points have non-zero and zero third coordinate, $\{[0,0,1]^{\text{tr}}, [1,0,1]^{\text{tr}}, [0,1,1]^{\text{tr}}, [1,1,1]^{\text{tr}}\}$ and $\{[1,0,0]^{\text{tr}}, [0,1,0]^{\text{tr}}, [1,1,0]^{\text{tr}}\}$, respectively. The latter correspond to the pencils of parallel lines of $\mathbb{A}^2(2)$ as follows, where the finite lines of $\mathbb{P}^2(2)$ are also indicated, the line at infinity being given as $z=0$; see Table 4:

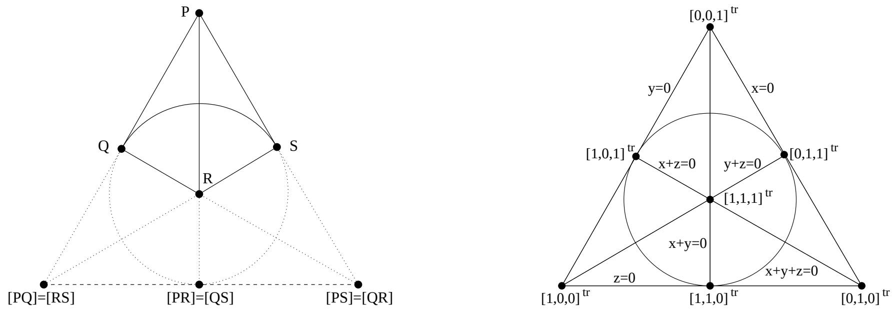
pencil of parallel lines	$[a,b]$	ideal point	finite projective lines
$x+c=0,\quad c\in\mathbb{F}_2$	$[1,0]$	$[0,1,0]^{\text{tr}}$	$x+cz=0,\quad c\in\mathbb{F}_2$
$y+c=0,\quad c\in\mathbb{F}_2$	$[0,1]$	$[1,0,0]^{\text{tr}}$	$y+cz=0,\quad c\in\mathbb{F}_2$
$x+y+c=0,\quad c\in\mathbb{F}_2$	$[1,1]$	$[1,1,0]^{\text{tr}}$	$x+y+cz=0,\quad c\in\mathbb{F}_2$

(2.8) Finite planes. a) Let $\mathcal{P}$ be a finite projective plane. Then all lines of $\mathcal{P}$ contain the same number $q+1\geq 3$ of points, where $q\geq 2$ is called the **order** of $\mathcal{P}$. Given a point P, taking a line m such that $P\notin m$ yields a well-defined map $\{l \text{ line}; P\in l\}\to\{Q \text{ point}; Q\in m\}\colon l\mapsto Q_l$, where $Q_l\in l\cap m$, having the inverse $\{Q \text{ point}; Q\in m\}\to\{l \text{ line}; P\in l\}\colon Q\mapsto PQ$, hence is a bijection. Thus there are $q+1$ lines containing P.

Letting $v\in\mathbb{N}$ and $b\in\mathbb{N}$ be the number of points and of lines, respectively, double counting the incident point-line pairs yields $v(q+1)=b(q+1)$, that is $v=b$. Moreover, we get $v=b=\frac{\binom{v}{2}}{\binom{q+1}{2}}=\frac{v(v-1)}{q(q+1)}$, implying $v=q^2+q+1$. Hence we recover in generality the figures found for projective planes over finite fields.

Hence if $v\in\mathbb{N}$ is not of the above form than there cannot possibly be a projective plane having precisely v points. Conversely, given $q\geq 2$ we may ask for the

Table 4: The Fano plane.



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existence of a projective plane having order q: For any prime power q there is the projective plane $\mathbb{P}^2(q) = \mathbb{P}^2(\mathbb{F}_q)$ over $\mathbb{F}_q$. The **prime power conjecture** says that the order of a projective plane necessarily is a prime power, but apart from the cases excluded by the Bruck-Ryser Theorem (2.9), for example $q = 6$ and $q = 14$, and the particular case $q = 10$ excluded by a computer-based search [Lam, 1991], this is widely open.

q	2	3	4	5	6	7	8	9	10	11	12	13	14	15
v	7	13	21	31	43	57	73	91	111	133	157	183	211	241
exists	+	+	+	+	−	+	+	+	−	+	?	+	−	?

Up to isomorphism, projective planes of order $q \in \{2, \ldots, 5, 7, 8\}$ are unique, which hence are those over the corresponding finite fields, while there are 4 projective planes of order $q = 9$ [Lam, 1991], and there are at least 2 projective planes of order p^2, whenever p is an odd prime.

b) Let $\mathcal{A}$ be a finite affine plane. Then going over to the completion $\widehat{\mathcal{A}}$ shows that all lines of $\mathcal{A}$ contain the same number $q \geq 2$ of points, called the **order** of $\mathcal{A}$. Moreover, any point lies on $q+1$ lines, there are $q+1$ pencils of parallel lines, each of which consists of q lines, and there are $v = (q^2 + q + 1) - (q + 1) = q^2$ points and $(q^2 + q + 1) - 1 = q^2 + q$ lines. Hence we recover in generality the figures found for affine planes over finite fields.

(2.9) Theorem: Bruck-Ryser [1949]. Let $\mathcal{P}$ be a finite projective plane of order $q \equiv \{1, 2\} \pmod 4$. Then we have $q = a^2 + b^2$ for some $a, b \in \mathbb{Z}$.

Proof. Omitted. ♯

3 Fano

(3.1) Projective spaces over skew fields. Let $\mathbb{F}$ be a skew field, and for $d \in \mathbb{N}_0$ we consider the $(d+1)$-dimensional (left) $\mathbb{F}$-vector space $V := \mathbb{F}^{(d+1)\times 1}$. Then the **affine space** $\mathbb{A}^{d+1}(\mathbb{F})$ **of dimension** $d+1$ **over** $\mathbb{F}$ is the geometry of rank $d+1$ whose elements are the additive cosets of the $\mathbb{F}$-subspaces of V, in other words the sets of solutions in V of all systems of linear equations. The incidence relation is given by set-theoretic inclusion, and the type of an element is given by the $\mathbb{F}$-dimension of the underlying $\mathbb{F}$-subspace, in other words by the $\mathbb{F}$-dimension of the $\mathbb{F}$-subspace of solutions of an associated homogeneous system of linear equations; in particular, V is the set of points of $\mathbb{A}^{d+1}(\mathbb{F})$. Hence for $d = 1$ we recover the affine plane $\mathbb{A}^2(\mathbb{F})$ over $\mathbb{F}$, and for $d = 0$ we get the **affine line** $\mathbb{A}^1(\mathbb{F})$ **over** $\mathbb{F}$, whose elements are just the points and a single line.

The **projective space** $\mathbb{P}^d(\mathbb{F})$ **of dimension** $d \geq 1$ **over** $\mathbb{F}$ is the rank 2 subgeometry of $\mathbb{A}^{d+1}(\mathbb{F})$, whose points and lines are the affine lines and planes containing 0, respectively. Hence for $d = 1$ we get the **projective line** $\mathbb{P}^1(\mathbb{F})$ **over** $\mathbb{F}$ whose elements are just the points and a single line; note that for $d = 0$ we just get a single point, a rank 1 geometry.

Thus the points of $\mathbb{P}^d(\mathbb{F})$ are the 1-dimensional $\mathbb{F}$-subspaces $\langle v\rangle_{\mathbb{F}} \leq V$, where $0 \neq v = [x_1, \ldots, x_d, x_0]^{\text{tr}} \in V$ are called associated **homogeneous coordinates**; the latter are only unique up to non-zero scalars. The lines are the 2-dimensional $\mathbb{F}$-subspaces $\langle v, v'\rangle_{\mathbb{F}} = \langle\langle v\rangle_{\mathbb{F}}, \langle v'\rangle_{\mathbb{F}}\rangle_{\mathbb{F}} \leq V$, where $\{v, v'\} \subseteq V$ is $\mathbb{F}$-linearly independent, the incidence relation being set-theoretic inclusion. Hence the points contained in the line $\langle v, v'\rangle_{\mathbb{F}}$ are given by $\{\langle av + bv'\rangle_{\mathbb{F}}; 0 \neq [a, b] \in \mathbb{F}^2\} = \{\langle cv + v'\rangle_{\mathbb{F}}; c \in \mathbb{F}\} \,\dot{\cup}\, \{\langle v\rangle_{\mathbb{F}}\}$; in particular $\mathbb{P}^d(q) := \mathbb{P}^d(\mathbb{F}_q)$ is finite of **order** q, that is any line contains precisely $q+1$ points. Then $\mathbb{P}^d(\mathbb{F})$ indeed is a projective space, which fulfils (P4) and is non-degenerate if and only if $d \geq 2$:

i) Let $\langle v\rangle_{\mathbb{F}} \neq \langle v'\rangle_{\mathbb{F}}$, then $\{v, v'\} \subseteq V$ is $\mathbb{F}$-linearly independent, hence $\langle v, v'\rangle_{\mathbb{F}} \leq V$ is the unique 2-dimensional $\mathbb{F}$-subspace containing $\{v, v'\}$, showing (P1).

ii) Let $\langle v\rangle_{\mathbb{F}}$, $\langle v'\rangle_{\mathbb{F}}$, $\langle w\rangle_{\mathbb{F}}$, $\langle w'\rangle_{\mathbb{F}}$ be pairwise distinct points such that $\langle v, v'\rangle_{\mathbb{F}} \neq \langle w, w'\rangle_{\mathbb{F}}$ and $\langle v, v'\rangle_{\mathbb{F}} \cap \langle w, w'\rangle_{\mathbb{F}} \neq \{0\}$. Hence we have $\dim_{\mathbb{F}}(\langle v, v'\rangle_{\mathbb{F}} \cap \langle w, w'\rangle_{\mathbb{F}}) = 1$, from which we get $\dim_{\mathbb{F}}(\langle v, v', w, w'\rangle_{\mathbb{F}}) = \dim_{\mathbb{F}}(\langle v, v'\rangle_{\mathbb{F}}) + \dim_{\mathbb{F}}(\langle w, w'\rangle_{\mathbb{F}}) - \dim_{\mathbb{F}}(\langle v, v'\rangle_{\mathbb{F}} \cap \langle w, w'\rangle_{\mathbb{F}}) = 2 + 2 - 1 = 3$. This yields $\dim_{\mathbb{F}}(\langle v, w\rangle_{\mathbb{F}} \cap \langle v', w'\rangle_{\mathbb{F}}) = \dim_{\mathbb{F}}(\langle v, w\rangle_{\mathbb{F}}) + \dim_{\mathbb{F}}(\langle v', w'\rangle_{\mathbb{F}}) - \dim_{\mathbb{F}}(\langle v, v', w, w'\rangle_{\mathbb{F}}) = 2 + 2 - 3 = 1$, hence we infer $\langle v, w\rangle_{\mathbb{F}} \neq \langle v', w'\rangle_{\mathbb{F}}$ and $\langle v, w\rangle_{\mathbb{F}} \cap \langle v', w'\rangle_{\mathbb{F}} \neq \{0\}$, showing (P2').

iii) If $d \geq 2$, hence $d + 1 \geq 3$, let $\{v, v', v''\} \subseteq V$ be $\mathbb{F}$-linearly independent, hence $\langle v, v'\rangle_{\mathbb{F}} \leq V$ is 2-dimensional and $v'' \notin \langle v, v'\rangle_{\mathbb{F}}$, showing (P3).

iv) Let $\{v, v'\} \subseteq V$ be $\mathbb{F}$-linearly independent, then the line $\langle v, v'\rangle_{\mathbb{F}}$ contains the points $\langle v\rangle_{\mathbb{F}}$, $\langle v'\rangle_{\mathbb{F}}$ and $\langle v + v'\rangle_{\mathbb{F}}$, showing (P4). ♯

Moreover, $\mathbb{P}^d(\mathbb{F})$ is a projective plane if and only if $d = 2$: If $d = 2$ then for any lines $\langle v, v'\rangle_{\mathbb{F}} \neq \langle w, w'\rangle_{\mathbb{F}}$ we have $\dim_{\mathbb{F}}(\langle v, v'\rangle_{\mathbb{F}} \cap \langle w, w'\rangle_{\mathbb{F}}) = \dim_{\mathbb{F}}(\langle v, v'\rangle_{\mathbb{F}}) + \dim_{\mathbb{F}}(\langle w, w'\rangle_{\mathbb{F}}) - \dim_{\mathbb{F}}(\langle v, v', w, w'\rangle_{\mathbb{F}}) = 2 + 2 - 3 = 1$, implying $\langle v, v'\rangle_{\mathbb{F}} \cap \langle w, w'\rangle_{\mathbb{F}} \neq \{0\}$, hence (P2) holds. If $d \geq 3$ then letting $\{v, v', w, w'\} \subseteq V$ be $\mathbb{F}$-

linearly independent, for the lines $\langle v, v' \rangle_{\mathbb{F}} \neq \langle w, w' \rangle_{\mathbb{F}}$ we have $\langle v, v' \rangle_{\mathbb{F}} \cap \langle w, w' \rangle_{\mathbb{F}} = \{0\}$, thus (P2) does not hold.

(3.2) Fano spaces. A non-degenerate projective space is called a **Fano space** or a **non-Fano space**, if either of the axioms (PF) and (PnF) is fulfilled:

(PF, PnF) Given a **planar** quadrangle $\{P, Q, R, S\}$, that is we have $PQ \not\parallel RS$, then letting $A \in PQ \cap RS$ and $B \in PR \cap QS$ and $C \in PS \cap QR$ the **diagonal** points $\{A, B, C\}$ are: **(PF)** collinear; **(PnF)** non-collinear.

Note that by (P2') in a planar quadrangle we have $PR \not\parallel QS$ and $PS \not\parallel QR$, and there always is a planar quadrangle: By (P3) let $\{X, Y, Z\}$ be non-collinear, then by (P4) there are $X' \in XZ$ and $Y' \in YZ$ such that $\{X', Y'\} \cap \{X, Y, Z\} = \emptyset$.

The points $\{A, B, C\}$ are pairwise distinct: Assume that $A = B$, then we have $A = B = P = Q = R = S \in PQ \cap PR \cap RS \cap QS$, a contradiction. Moreover, we have $\{P, Q, R, S\} \cap \{A, B, C\} = \emptyset$: Assume that $P = A$, then $P \in RS$, a contradiction. Thus $\{P, Q, R, S, A, B, C\}$ are pairwise distinct points.

The configuration consisting of the 7 points $\{P, Q, R, S, A, B, C\}$ and the 6 lines $\{PQ, PR, PS, QR, QS, RS\}$ is called the **complete quadrangle**. Here, a **configuration** is a geometry of rank 2 such that any line contains at least 2 points and any 2 distinct points lie on at most one line.

If (PF) holds, then $AB = AC = BC \notin \{PQ, PR, PS, QR, QS, RS\}$: Assume that $AB = PQ$, then $B = P \in PQ \cap PR$, a contradiction. Hence the lines $\{PQ, PR, PS, QR, QS, RS, AB\}$ are pairwise distinct. Thus the **Fano configuration**, see Table 5, consists of 7 points and lines each, where any line contains 3 points and any point lies on 3 lines, hence coincides with the Fano plane.

(3.3) Dualising the Fano axioms. This yields:

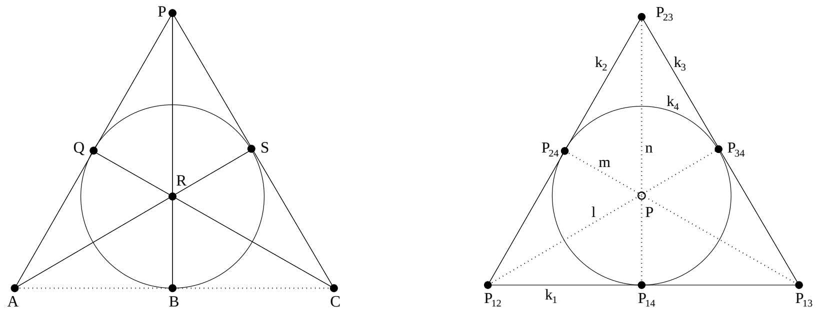
(PF*, PnF*) Given a **planar quadrilateral** $\{k_1, \ldots, k_4\}$, that is 4 pairwise non-parallel lines no 3 of which are concurrent, then letting $l := P_{12}P_{34}$ and $m := P_{13}P_{24}$ and $n := P_{14}P_{23}$, where $P_{ij} \in l_i \cap l_j$ for $i \neq j \in \{1, \ldots, 4\}$, the **diagonal** lines $\{l, m, n\}$ are: **(PF*)** concurrent; **(PnF*)** non-concurrent.

The lines $\{l, m, n\}$ are pairwise distinct: Assume that $l = m$, then we infer that $\{P_{12}, P_{13}, P_{24}, P_{34}\}$ are collinear, thus $l = m = k_1 = \ldots = k_4$, a contradiction. Moreover, we have $\{k_1, \ldots, k_4\} \cap \{l, m, n\} = \emptyset$: Assume that $l = k_1$, then $P_{34} \in k_1$, a contradiction. The configuration consisting of the 6 points $\{P_{12}, \ldots, P_{34}\}$ and the 7 lines $\{k_1, \ldots, k_4, l, m, n\}$ is called the **complete quadrilateral**.

If (PF*) holds, then letting $P \in l \cap m \cap n$ we have $P \notin \{P_{12}, \ldots, P_{34}\}$: Assume that $P = P_{12}$, then $m = PP_{13} = P_{12}P_{13} = k_1$, a contradiction. Thus the **dual Fano configuration**, see Table 5, also consists of 7 points and lines each, where any line contains 3 points and any point lies on 3 lines; indeed the dual Fano configuration is isomorphic to the Fano configuration.

Given a non-degenerate projective space, (PF) holds if and only if (PF*) holds, and (PnF) holds if and only if (PnF*) holds: The quadrangle $\{P_{12}, P_{13}, P_{24}, P_{34}\}$

Table 5: The Fano configuration and its dual.



yields $P_{14} \in P_{12}P_{13} \cap P_{24}P_{34} = k_1 \cap k_4$ and $P_{23} \in P_{12}P_{24} \cap P_{13}P_{34} = k_2 \cap k_3$ and $P := P_{12}P_{34} \cap P_{13}P_{24} = l \cap m$, hence since $n = P_{14}P_{23}$ we conclude that $\{l, m, n\}$ is concurrent if and only if $\{P_{14}, P_{23}, P\}$ is collinear; thus (PF) implies (PF*), and (PnF) implies (PnF*). Conversely, the quadrilateral $\{AP, CP, AR, CR\}$ yields $P \in AP \cap CP$ and $R \in AR \cap CR$, as well as $A \in AP \cap AR$ and $C \in CP \cap CR$, as well as $Q \in AP \cap CR$ and $S \in CP \cap AR$, hence since $B \in PR \cap QS$ we conclude that $\{A, B, C\}$ is collinear if and only if $\{PR, AC, QS\}$ is concurrent; thus (PF*) implies (PF), and (PnF*) implies (PnF).

(3.4) Theorem: Fano. Let $\mathbb{F}$ be a skew field and $d \geq 2$. Then $\mathbb{P}^d(\mathbb{F})$ is a Fano space if $\mathrm{char}(\mathbb{F}) = 2$, otherwise it is a non-Fano space.

Proof. Let $P = \langle p \rangle_{\mathbb{F}}$, $Q = \langle q \rangle_{\mathbb{F}}$, $R = \langle r \rangle_{\mathbb{F}}$, $S = \langle s \rangle_{\mathbb{F}}$, for suitable $0 \neq p, q, r, s \in \mathbb{F}^{(d+1)\times 1}$, fulfilling the assumptions of (PF). Then from $\{P, Q, R\}$ being non-collinear we infer that $\{p, q, r\}$ is $\mathbb{F}$-linearly independent. Since $PQ \not\parallel RS$ we have $\langle p, q \rangle_{\mathbb{F}} \cap \langle r, s \rangle_{\mathbb{F}} \neq \{0\}$, showing that $\dim_{\mathbb{F}}(\langle p, q, r, s \rangle_{\mathbb{F}}) = \dim_{\mathbb{F}}(\langle p, q \rangle_{\mathbb{F}}) + \dim_{\mathbb{F}}(\langle r, s \rangle_{\mathbb{F}}) - \dim_{\mathbb{F}}(\langle p, q \rangle_{\mathbb{F}} \cap \langle r, s \rangle_{\mathbb{F}}) = 2 + 2 - 1 = 3$, thus $s \in \langle p, q, r \rangle_{\mathbb{F}}$. Since $\{P, Q, R, S\}$ is a quadrangle there are $0 \neq \alpha, \beta, \gamma \in \mathbb{F}$ such that $s = \alpha p + \beta q + \gamma r$, where we hence may assume that $\alpha = \beta = \gamma = 1$, that is $s = p + q + r$.

Hence we have $A = \langle p, q \rangle_{\mathbb{F}} \cap \langle r, s \rangle_{\mathbb{F}} = \langle p, q \rangle_{\mathbb{F}} \cap \langle r, p + q + r \rangle_{\mathbb{F}} = \langle p + q \rangle_{\mathbb{F}}$ and $B = \langle p, r \rangle_{\mathbb{F}} \cap \langle q, s \rangle_{\mathbb{F}} = \langle p, r \rangle_{\mathbb{F}} \cap \langle q, p+q+r \rangle_{\mathbb{F}} = \langle p+r \rangle_{\mathbb{F}}$ and $C = \langle p, s \rangle_{\mathbb{F}} \cap \langle q, r \rangle_{\mathbb{F}} = \langle p, p + q + r \rangle_{\mathbb{F}} \cap \langle q, r \rangle_{\mathbb{F}} = \langle q + r \rangle_{\mathbb{F}}$. Thus we have $AB = AC = BC$ if and only if $\dim_{\mathbb{F}}(\langle p + q, p + r, q + r \rangle_{\mathbb{F}}) = 2$. Gaussian elimination yields

$$\begin{bmatrix} 1 & 1 & . \\ 1 & . & 1 \\ . & 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & . \\ . & 1 & -1 \\ . & 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & . \\ . & 1 & -1 \\ . & . & 2 \end{bmatrix} \in \mathbb{F}^{3\times 3},$$

thus we have $\dim_{\mathbb{F}}(\langle p + q, p + r, q + r \rangle_{\mathbb{F}}) = 2$ if and only if $2 = 0 \in \mathbb{F}$. ♯

4 Pappus

(4.1) Pappian spaces. A non-degenerate projective space is called **Pappian** if the following axiom (PP) is fulfilled:

(PP) Given pairwise distinct collinear points $\{P, Q, R\}$ and $\{P', Q', R'\}$ such that $PQ \not\parallel P'Q'$ and $S \in PQ \cap P'Q'$ fulfils $S \notin \{P, Q, R, P', Q', R'\}$, then letting $A \in PQ' \cap QP'$ and $B \in PR' \cap RP'$ and $C \in QR' \cap RQ'$ the set $\{A, B, C\}$ is collinear.

Hence we have $P', Q', R' \notin PQ$ and $P, Q, R \notin P'Q'$, thus we have $\{P, Q, R\} \cap \{P', Q', R'\} = \emptyset$. Since $PQ \not\parallel P'Q'$, axiom (P2') implies that $PQ' \not\parallel PQ'$, and similarly $PR' \not\parallel RP'$ and $QR' \not\parallel RQ'$, thus $\{A, B, C\}$ are uniquely defined. Moreover, $\{A, B, C\}$ are pairwise distinct: Assume that $A = B$, then $PR' = PQ'$, hence $P \in P'Q'$, a contradiction. Finally, we have $\{A, B, C\} \cap \{P, Q, R, P', Q', R'\} = \emptyset$: Assume that $A = P$, then $QP' = PQ$, hence $P' \in PQ$, a contradiction; assume that $A = R$, then $PQ' = PR$, hence $Q' \in PQ$, a contradiction. Thus $\{P, Q, R, P', Q', R', A, B, C\}$ are pairwise distinct points.

Moreover, the lines $\{PQ, P'Q', PQ', QP', PR', RP', QR', RQ', AB\}$ are pairwise distinct as well: Assume that $PQ' = PR'$, then $P \in P'Q'$, a contradiction; assume that $PQ' = QR'$, then $PQ = P'Q'$, a contradiction; assume that $PQ = PQ'$, then $Q' \in PQ$, a contradiction; assume that $PQ = AB$, then $QP' = QA = PQ$, hence $P' \in PQ$, a contradiction; assume that $PQ' = AB$, then $PR' = PB = PQ'$, a contradiction.

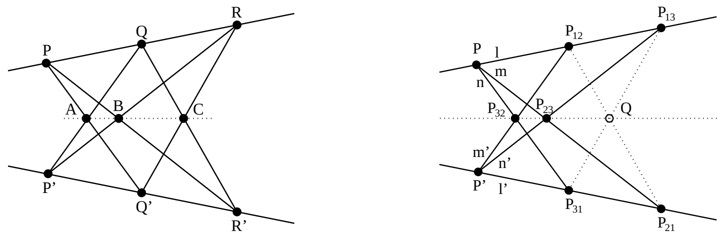
Thus the **Pappus configuration**, see Table 6, consists of 9 points and lines each, where any line contains 3 points and any point lies on 3 lines. In other words, (PP) says that if a **hexagon** $PQ'RP'QR'$ is inscribed on intersecting lines, then the pairs $\{PQ', P'Q\}$, $\{Q'R, QR'\}$, $\{RP', R'P\}$ of opposite **sides** meet in collinear points.

(4.2) Dualising the Pappus axiom. This yields:

(PP*) Given pairwise distinct concurrent lines $\{l, m, n\}$ and $\{l', m', n'\}$ such that letting $P \in l \cap m \cap n$ and $P' \in l' \cap m' \cap n'$ we have $P \neq P'$ and $PP' \notin \{l, m, n, l', m', n'\}$, and $l \not\parallel m', n'$ and $m \not\parallel l', n'$ and $n \not\parallel l', m'$, then letting $P_{12} \in l \cap m'$ and $P_{21} \in m \cap l'$, as well as $P_{13} \in l \cap n'$ and $P_{31} \in n \cap l'$, as well as $P_{23} \in m \cap n'$ and $P_{32} \in n \cap m'$ the set $\{P_{12}P_{21}, P_{13}P_{31}, P_{23}P_{32}\}$ is concurrent.

Hence we have $P \notin l', m', n'$ and $P' \notin l, m, n$: Assume that $P \in l'$, then $PP' = l'$, a contradiction. Thus we have $\{l, m, n\} \cap \{l', m', n'\} = \emptyset$. Moreover we have $\{P_{12}, P_{21}, P_{13}, P_{31}, P_{23}, P_{32}\} \cap \{P, P'\} = \emptyset$: Assume that $P_{12} = P$, then $PP' = m'$, a contradiction. The points $\{P_{12}, P_{21}, P_{13}, P_{31}, P_{23}, P_{32}\}$ are pairwise distinct: Assume that $P_{12} = P_{13}$, then $m' = n'$, a contradiction; assume that $P_{12} = P_{23}$, then $l = m$, a contradiction. Hence in particular $P_{12}P_{21}$ and $P_{13}P_{31}$ and $P_{23}P_{32}$ are well-defined. Moreover we have $\{P_{12}P_{21}, P_{13}P_{31}, P_{23}P_{32}\} \cap \{l, m, n, l', m', n'\} = \emptyset$: Assume that $P_{12}P_{21} = l$, then $P_{21} = P \in l \cap m$, a contradiction; assume that $P_{12}P_{21} = n$, then $P_{21} = P \in m \cap n$, a contradiction.

Table 6: The Pappus configuration and its dual.



Thus $\{l, m, n, l', m', n', P_{12}P_{21}, P_{13}P_{31}, P_{23}P_{32}\}$ are pairwise distinct lines.

Since $P_{21}P = m \nparallel n' = P_{13}P'$, axiom (P2') implies that $l = P_{13}P \nparallel P_{21}P' = l'$, and similarly $m \nparallel m'$ and $n \nparallel n'$. Thus since $P_{12}P_{13} = l \nparallel l' = P_{21}P_{31}$, axiom (P2') implies that $P_{12}P_{21} \nparallel P_{13}P_{31}$ anyway, and similarly $P_{12}P_{21} \nparallel P_{23}P_{32}$ and $P_{13}P_{31} \nparallel P_{23}P_{32}$. Let $Q \in P_{12}P_{21} \cap P_{13}P_{31} \cap P_{23}P_{32}$. Then we have $Q \notin \{P_{12}, P_{21}, P_{13}, P_{31}, P_{23}, P_{32}, P, P'\}$: Assume that $Q = P_{12}$, then $P_{31} \in P_{13}Q = P_{13}P_{12} = l$, hence $P_{31} \in l \cap n$, a contradiction; assume that $Q = P$, then $P_{21} \in P_{12}Q = P_{12}P = l$, hence $P_{21} = P \in l \cap m$, a contradiction. Thus $\{P_{12}, P_{21}, P_{13}, P_{31}, P_{23}, P_{32}, P, P', Q\}$ are pairwise distinct points.

Thus the **dual Pappus configuration**, see Table 6, also consists of 9 points and lines each, where any line contains 3 points and any point lies on 3 lines; indeed the dual Pappus configuration is isomorphic to the Pappus configuration.

Given a non-degenerate projective space, (PP) holds if and only if (PP*) holds: If (PP) holds, then considering the lines $PP_{12} = PP_{13}$ and $P'P_{31} = P'P_{21}$ shows that letting $Q \in P_{12}P_{21} \cap P_{31}P_{13}$ the points $\{P_{23}, P_{32}, Q\}$ are collinear, thus $\{P_{12}P_{21}, P_{31}P_{13}, P_{23}P_{32}\}$ is concurrent. If (PP*) holds, then considering $l := PQ = PR$ and $m := PR'$ and $n := PQ'$ and $l' := P'Q' = P'R'$ and $m' := P'Q$ and $n' := P'R$ yields $A \in n \cap m'$ and $B \in m \cap n'$, thus $\{AB, QR', RQ'\}$ is concurrent, hence $C \in QR' \cap RQ'$ yields that $\{A, B, C\}$ is collinear.

(4.3) Theorem: Pappus. Let $\mathbb{F}$ be a skew field and $d \geq 2$. Then $\mathbb{P}^d(\mathbb{F})$ is Pappian if and only if $\mathbb{F}$ is a field.

Proof. Let $P = \langle p \rangle_{\mathbb{F}}$, $Q = \langle q \rangle_{\mathbb{F}}$, $R = \langle r \rangle_{\mathbb{F}}$, and $P' = \langle p' \rangle_{\mathbb{F}}$, $Q' = \langle q' \rangle_{\mathbb{F}}$, $R' = \langle r' \rangle_{\mathbb{F}}$, and $S = \langle s \rangle_{\mathbb{F}}$, for suitable $0 \neq p, q, r, p', q', r', s \in \mathbb{F}^{(d+1)\times 1}$, fulfilling the assumptions of (PP). Hence we have $l = \langle s, p \rangle_{\mathbb{F}}$ and $l' = \langle s, p' \rangle_{\mathbb{F}}$, and since $P' \notin l$ we conclude that $\{p, p', s\}$ is $\mathbb{F}$-linearly independent. We may assume that $q = p + s$ and $r = \alpha p + s$ as well as $q' = p' + s$ and $r' = \beta p' + s$, where $\alpha, \beta \in \mathbb{F} \setminus \{0, 1\}$; note that $\{0, 1\}$ commute with all elements of $\mathbb{F}$ anyway.

We have $A = \langle p, q' \rangle_{\mathbb{F}} \cap \langle q, p' \rangle_{\mathbb{F}} = \langle p, p' + s \rangle_{\mathbb{F}} \cap \langle p + s, p' \rangle_{\mathbb{F}} = \langle p + p' + s \rangle_{\mathbb{F}}$ and

$B = \langle p, r' \rangle_{\mathbb{F}} \cap \langle r, p' \rangle_{\mathbb{F}} = \langle p, \beta p' + s \rangle_{\mathbb{F}} \cap \langle \alpha p + s, p' \rangle_{\mathbb{F}} = \langle \alpha p + \beta p' + s \rangle_{\mathbb{F}}$. Gaussian elimination, letting $\gamma := (1-\alpha)(1-\beta)^{-1} \in \mathbb{F}$, yields

$$\left[\begin{array}{ccc|ccc} 1 & . & 1 & 1 & . & 1 \\ . & \beta & 1 & . & \beta & 1 \\ \hline \alpha & . & 1 & . & . & . \\ . & 1 & 1 & . & . & . \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & . & 1 & 1 & . & 1 \\ . & 1 & 1 & . & . & . \\ . & . & 1-\beta & . & \beta & 1 \\ \hline . & . & . & \alpha & \gamma\beta & \alpha+\gamma \end{array}\right] \in \mathbb{F}^{4 \times 6},$$

thus we have $C = \langle q, r' \rangle_{\mathbb{F}} \cap \langle r, q' \rangle_{\mathbb{F}} = \langle p + s, \beta p' + s \rangle_{\mathbb{F}} \cap \langle \alpha p + s, p' + s \rangle_{\mathbb{F}} = \langle \alpha p + \gamma\beta p' + (\alpha+\gamma) s \rangle_{\mathbb{F}}$. Hence we have $C \in AB$ if and only if $\alpha p + \gamma\beta p' + (\alpha+\gamma) s \in \langle p + p' + s, \alpha p + \beta p' + s \rangle_{\mathbb{F}}$. Again Gaussian elimination yields

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ \alpha & \beta & 1 \\ \hline \alpha & \gamma\beta & \alpha+\gamma \end{array}\right] \mapsto \left[\begin{array}{ccc} 1 & 1 & 1 \\ \alpha-1 & \beta-1 & . \\ \hline \gamma & 1 & . \end{array}\right] \mapsto \left[\begin{array}{ccc} 1 & 1 & 1 \\ (1-\beta)^{-1}(\alpha-1) & 1 & . \\ \hline (1-\alpha)(1-\beta)^{-1} & 1 & . \end{array}\right] \in \mathbb{F}^{4 \times 6},$$

showing that $C \in AB$ if and only if $(1-\beta)^{-1}(\alpha-1) = (1-\alpha)(1-\beta)^{-1}$, or equivalently $(\alpha-1)(1-\beta) = (1-\beta)(1-\alpha)$, that is $\alpha\beta = \beta\alpha$. Hence (PP) holds if and only if $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{F} \setminus \{0, 1\}$, that is $\mathbb{F}$ is commutative. ♯

5 Desargues

(5.1) Desarguesian spaces. A non-degenerate projective space is called **Desarguesian** if the following axiom (PD) is fulfilled:

(PD) Given non-collinear points $\{P, Q, R\}$ and $\{P', Q', R'\}$ such that $P \neq P'$ and $Q \neq Q'$ and $R \neq R'$, and the lines $\{PP', QQ', RR'\}$ are pairwise distinct and concurrent, then letting $A \in PQ \cap P'Q'$ and $B \in QR \cap Q'R'$ and $C \in PR \cap P'R'$ the set $\{A, B, C\}$ is collinear.

Note that $PP' \neq QQ'$ is equivalent to $PQ \neq P'Q'$, similarly $QQ' \neq RR'$ is equivalent to $QR \neq Q'R'$, and $PP' \neq RR'$ is equivalent to $PR \neq P'R'$. Since $PP' \not\parallel QQ'$, axiom (P2') implies that $PQ \not\parallel P'Q'$, and similarly $QR \not\parallel Q'R'$ and $PR \not\parallel P'R'$, thus $\{A, B, C\}$ are uniquely defined. Moreover, $\{A, B, C\}$ are pairwise distinct: Assume that $A = B$, then $A = B \in (PQ \cap QR) \cap (P'Q' \cap Q'R')$ implies $Q = A = B = Q'$, a contradiction.

Let $S \in PP' \cap QQ' \cap RR'$. We may assume that $S \notin \{P, Q, R, P', Q', R'\}$: Assume that $S = P$, then $Q' \in QS = PQ$ and $R' \in RS = PR$, hence $A = Q' \in PQ \cap P'Q'$ and $C = R' \in PR \cap P'R'$, showing that $\{A, B, C\} \subseteq Q'R'$ in this case is collinear anyway.

Now we infer that $\{P, Q, R, S\}$ and $\{P', Q', R', S\}$ are quadrangles: Assume that $S \in PQ$, then $P' \in PS = PQ$ and $Q' \in QS = PQ$, thus $PQ = P'Q'$, a contradiction. Moreover, we have $\{P, Q, R\} \cap \{P', Q', R'\} = \emptyset$: Assume that $P = Q'$, then $S \in QQ' = PQ$, a contradiction. Finally, we have $\{A, B, C\} \cap \{P, Q, R, P', Q', R', S\} = \emptyset$: Assume that $A = P$, then $S \in PP' = P'Q'$, a contradiction; assume that $A = R$, then $R \in PQ$, a contradiction;

assume that $A = S$, then $S \in PQ$, a contradiction. Thus we conclude that $\{P, Q, R, P', Q', R', S, A, B, C\}$ are pairwise distinct points.

The lines $\{PQ, QR, PR, P'Q', Q'R', P'R', PP', QQ', RR', AB\}$ are pairwise distinct as well: Assume that $PQ = Q'R'$, then $S \in QQ' = PQ$, a contradiction; assume that $PQ \in \{PP', RR'\}$, then $S \in PQ$, a contradiction; assume that $PQ = AB$, then $PQ = QB = QR$, a contradiction; assume that $PP' = AB$, then $S \in PP' = PA = PQ$, a contradiction.

Thus the **Desargues configuration**, see Table 7, consists of 10 points and lines each, where any line contains 3 points and any point lies on 3 lines. In other words, (PD) says that if triangles $\{P, Q, R\}$ and $\{P', Q', R'\}$ are **centrally perspective** with respect to the point $S \in PP' \cap QQ' \cap RR'$, then they are **axially perspective** with respect to the line $AB = BC = AC$.

(5.2) Dualising the Desargues axiom. This yields:

(PD*) Given pairwise non-parallel, non-concurrent lines $\{l, m, n\}$, $\{l', m', n'\}$, such that $l \not\parallel l'$ and $m \not\parallel m'$ and $n \not\parallel n'$, where letting $L \in l \cap l'$ and $M \in m \cap m'$ and $N \in n \cap n'$ the points $\{L, M, N\}$ are pairwise distinct and collinear, then letting $X \in l \cap m$ and $Y \in l \cap n$ and $Z \in m \cap n$ as well as $X' \in l' \cap m'$ and $Y' \in l' \cap n'$ and $Z' \in m' \cap n'$ the lines $\{XX', YY', ZZ'\}$ are concurrent.

Moreover, we have $X \neq X'$ and $Y \neq Y'$ and $Z \neq Z'$: Assume that $X = X'$, then $l \cap m = l' \cap m'$, hence $L = M \in (l \cap l') \cap (m \cap m')$, a contradiction. Hence the lines $a := XX'$ and $b := YY'$ and $c := ZZ'$ are well-defined. Moreover, $\{a, b, c\}$ are pairwise distinct: Assume that $a = b$, then $a = b = XX' = YY' = XY = X'Y' = l = l'$, a contradiction. Note that since $l \not\parallel l'$, axiom (P2') implies that $a \not\parallel b$ anyway, and similarly $a \not\parallel c$ and $b \not\parallel c$.

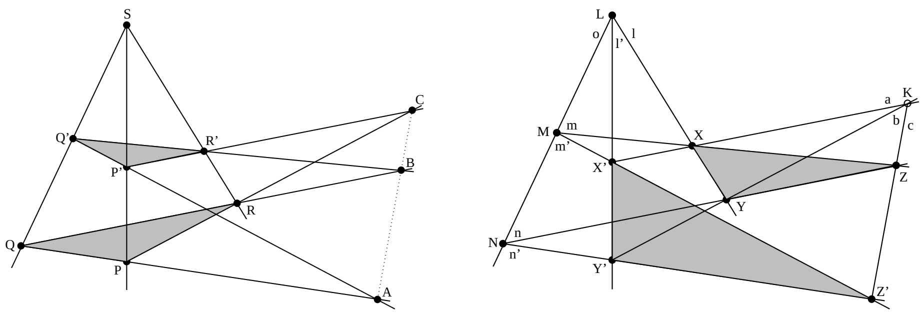
Let $o := LM = LN = MN$. We may assume that $o \notin \{l, m, n, l', m', n'\}$: Assume that $o = l$, then we have $M = X \in m \cap o = l \cap m$ and $N = Y \in n \cap o = l \cap n$, implying $a = XX' = X'Z'$ and $b = YY' = Y'Z'$, hence $Z' \in a \cap b \cap c$, showing that in this case $\{a, b, c\}$ is concurrent anyway.

Then we have $X, Y, Z, X', Y', Z' \notin o$: Assume that $X \in o$, then $X = L \in l \cap o$ and $X = M \in m \cap o$, a contradiction. Thus we have $\{l, m, n\} \cap \{l', m', n'\} = \emptyset$: Assume that $l = m'$, then $X' \in l' \cap m' = l \cap l' \subseteq o$, a contradiction. Moreover, we have $\{a, b, c\} \cap \{l, m, n, l', m', n', o\} = \emptyset$: Assume that $a = l$, then $X' \in l' \cap a = l \cap l' \subseteq o$, a contradiction; assume that $a = n$, then $X \in n = YZ$, a contradiction; assume that $a = o$, then $X \in o$, a contradiction. Thus we conclude that $\{l, m, n, l', m', n', o, a, b, c\}$ are pairwise distinct lines.

Then $\{X, Y, Z, X', Y', Z', L, M, N, K\}$ are pairwise distinct as well: Assume that $X = Y'$, then $X = Y' \in l \cap l' \subseteq o$, a contradiction; assume that $X \in \{L, N\}$, then $X \in o$, a contradiction; assume that $X = K$, then $X = Y \in l \cap m = l \cap b$, a contradiction; assume that $L = K$, then $l = XL = XK = a$, a contradiction.

Thus the **dual Desargues configuration**, see Table 7, also consists of 10 points and lines each, where any line contains 3 points and any point lies on 3 lines;

Table 7: The Desargues configuration and its dual.



indeed the dual Desargues configuration is isomorphic to the Desargues configuration. In other words, (PD*) says that if triangles $\{X, Y, Z\}$ and $\{X', Y', Z'\}$ are axially perspective with respect to the line $o = LM = LN = MN$, then they are centrally perspective with respect to the point $K \in a \cap b \cap c$; note that this is the converse of the implication in (PD).

A non-degenerate projective space fulfils (PD) if and only if it fulfils (PD*): If (PD) holds, then considering the triangles $\{X, X', M\}$ and $\{Y, Y', N\}$, we conclude that $\{Z, Z', K\}$ is collinear, thus $\{a, b, c\}$ is concurrent. Conversely, if (PD*) holds, then considering the triangles $\{P, P'A\}$ and $\{R, R', B\}$, we conclude that $\{PR, P'R', AB\}$ is concurrent, thus $\{A, B, C\}$ is collinear.

(5.3) Theorem: Desargues. Let $\mathbb{F}$ be a skew field and $d \geq 2$. Then $\mathbb{P}^d(\mathbb{F})$ is Desarguesian.

Proof. Let $P = \langle p \rangle_\mathbb{F}$, $Q = \langle q \rangle_\mathbb{F}$, $R = \langle r \rangle_\mathbb{F}$, and $P' = \langle p' \rangle_\mathbb{F}$, $Q' = \langle q' \rangle_\mathbb{F}$, $R' = \langle r' \rangle_\mathbb{F}$, and $S = \langle s \rangle_\mathbb{F}$, for suitable $0 \neq p, q, r, p', q', r', s \in \mathbb{F}^{(d+1)\times 1}$, fulfilling the assumptions of (PD). Then from $\{P, Q, R\}$ being non-collinear we infer that $\{p, q, r\}$ is $\mathbb{F}$-linearly independent. We may assume that $\{P, Q, R, S\}$ and $\{P', Q', R', S\}$ are quadrangles, hence $\{P, P', S\}$ and $\{Q, Q', S\}$ and $\{R, R', S\}$ are pairwise distinct and collinear.

If $s \notin \langle p, q, r \rangle_\mathbb{F}$, then $\{p, q, r, s\}$ is $\mathbb{F}$-linearly independent. Hence may assume that $p' = p + s$ and $q' = q + s$ and $r' = r + s$. Thus we have $A = \langle p, q \rangle_\mathbb{F} \cap \langle p', q' \rangle_\mathbb{F} = \langle p, q \rangle_\mathbb{F} \cap \langle p + s, q + s \rangle_\mathbb{F} = \langle p - q \rangle_\mathbb{F}$ and $B = \langle q, r \rangle_\mathbb{F} \cap \langle q', r' \rangle_\mathbb{F} = \langle q, r \rangle_\mathbb{F} \cap \langle q + s, r + s \rangle_\mathbb{F} = \langle q - r \rangle_\mathbb{F}$ and $C = \langle p, r \rangle_\mathbb{F} \cap \langle p', r' \rangle_\mathbb{F} = \langle p, r \rangle_\mathbb{F} \cap \langle p + s, r + s \rangle_\mathbb{F} = \langle p - r \rangle_\mathbb{F}$. Since $(p - q) + (q - r) = p - r$ we infer that $AB = BC = AC = \langle p - q, q - r \rangle_\mathbb{F}$.

If $s \in \langle p, q, r \rangle_\mathbb{F}$, then since $\{P, Q, R, S\}$ is a quadrangle we may assume that $s = p+q+r$. Thus there are $0 \neq \alpha, \beta, \gamma \in \mathbb{F}$ such that $p' = \alpha p + s = (\alpha+1)p + q + r$ and $q' = \beta q + s = p + (\beta+1)q + r$ and $r' = \gamma r + s = p + q + (\gamma+1)r$. Hence we have

$A = \langle p,q\rangle_{\mathbb{F}} \cap \langle p',q'\rangle_{\mathbb{F}} = \langle p,q\rangle_{\mathbb{F}} \cap \langle (\alpha+1)p+q+r, p+(\beta+1)q+r\rangle_{\mathbb{F}} = \langle \alpha p - \beta q\rangle_{\mathbb{F}}$ and $B = \langle q,r\rangle_{\mathbb{F}} \cap \langle q',r'\rangle_{\mathbb{F}} = \langle q,r\rangle_{\mathbb{F}} \cap \langle p+(\beta+1)q+r, p+q+(\gamma+1)r\rangle_{\mathbb{F}} = \langle \beta q - \gamma r\rangle_{\mathbb{F}}$ and $C = \langle p,r\rangle_{\mathbb{F}} \cap \langle p',r'\rangle_{\mathbb{F}} = \langle p,r\rangle_{\mathbb{F}} \cap \langle (\alpha+1)p+q+, p+q+(\gamma+1)r\rangle_{\mathbb{F}} = \langle \alpha p - \gamma r\rangle_{\mathbb{F}}$. Since $(\alpha p - \beta q) + (\beta q - \gamma r) = \alpha p - \gamma r$ we infer that $AB = BC = AC = \langle \alpha p - \beta q, \beta q - \gamma r\rangle_{\mathbb{F}}$, thus $\{A, B, C\}$ is collinear. ♯

(5.4) Example: Moulton planes [1902]. We modify the real affine plane $\mathbb{A}^2(\mathbb{R})$, yielding the the geometry $\mathcal{A}_\alpha$ of rank 2, where $\alpha > 0$, whose set of points still is $\mathbb{R}^{2\times 1}$, but whose lines are defined as follows: For $c \in \mathbb{R}$ we have $\mathcal{L}(1,0;-c) := \{[x,y]^{\text{tr}} \in \mathbb{R}^{2\times 1}; x = c\}$ and $\mathcal{L}(-a,1;-c) := \{[x,y]^{\text{tr}} \in \mathbb{R}^{2\times 1}; y = ax + c\}$ for $a \leq 0$, that is the affine lines which are vertical or have non-positive slope, as well as $\mathcal{L}_\alpha(-a,1;-c) := \{[x,y]^{\text{tr}} \in \mathbb{R}^{2\times 1}; y = ax + c; x \leq 0\} \,\dot{\cup}\, \{[x,y]^{\text{tr}} \in \mathbb{R}^{2\times 1}; y = \alpha a x + c; x > 0\}$ for $a > 0$, that is the affine lines of positive slope are 'bent at the vertical axis'. Thus still any line has a slope $a \in \mathbb{R} \,\dot{\cup}\, \{\infty\}$, and for $\alpha = 1$ we recover the real affine plane. The incidence relation being given by set-theoretic inclusion, $\mathcal{A}_\alpha$ indeed is an affine plane:

i) Let $[x,y]^{\text{tr}} \neq [x',y'] \in \mathbb{R}^{2\times 1}$ be points. We have to show that there is a unique line containing these points, where it suffices to assume that $x < 0$ and $x' > 0$, as well as $y' > y$. There is a unique line as desired if and only if the system of linear equations $[a,c] \cdot A = [y,y']$, where $A := \begin{bmatrix} x & \alpha x' \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ has a unique solution. This since $\det(A) = x - \alpha x' < 0$ indeed holds, showing (A1).

ii) Let l be a line and P be a point not lying on l. Then, since distinct lines intersect if and only if they have distinct slope, the line through P having the same slope as l is the unique one being parallel to l, showing (A2).

iii) The points $\{[0,0]^{\text{tr}}, [1,0]^{\text{tr}}, [0,1]^{\text{tr}}\}$ are non-collinear, showing (A3). ♯

The projective completion $\widehat{\mathcal{A}_\alpha}$ of $\mathcal{A}_\alpha$, being called a **Moulton plane**, is non-Desarguesian if and only if $\alpha \neq 1$: We choose the Desargues configuration to lie in the affine plane $\mathcal{A}_\alpha \cong (\widehat{\mathcal{A}_\alpha})_0$, where only the point C lies in the half-plane $x > 0$, and where PR and $P'R'$ have non-positive slope and AB has positive slope, showing that (PD) fails if and only if $\alpha \neq 1$; see Table 8.

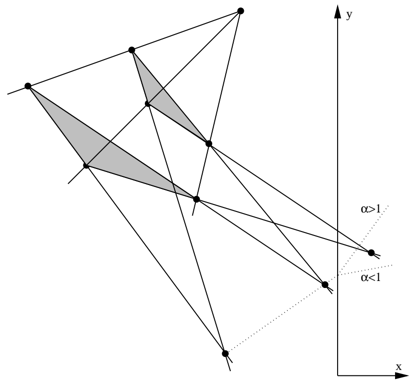
(5.5) Theorem: Hessenberg [1905]. Let $\mathcal{P}$ be a Pappian projective space. Then $\mathcal{P}$ is Desarguesian.

Proof. Omitted. ♯

(5.6) Remark. We consider the logical dependence of the axioms (PD), (PP), (PnF) and (PF) in the presence of (P1–4), that is for projective planes.

We have the following examples, where $\mathbb{H}$ are the **Hamilton quaternions**, and the **skew Laurent series field** $F := \mathbb{F}_4((X, \varphi_2))$ with respect to the Frobenius automorphism $\varphi_2 \in \text{Aut}(\mathbb{F}_4)$ is a non-commutative skew field of characteristic

Table 8: Desargues configuration in Moulton planes.



2. Moreover, $\widehat{\mathcal{A}}_\alpha$ is a Moulton plane with parameter $\alpha \neq 1$, and $\mathcal{P}$ is the **free projective plane** on $\mathbb{P}^2(2) \,\dot\cup\, \{\infty\}$:

	(PD)	(PP)	(PnF)	(PF)
$\mathbb{P}^2(\mathbb{R})$	$+$	$+$	$+$	$-$
$\mathbb{P}^2(\mathbb{H})$	$+$	$-$	$+$	$-$
$\mathbb{P}^2(2)$	$+$	$+$	$-$	$+$
$\mathbb{P}^2(F)$	$+$	$-$	$-$	$+$
$\widehat{\mathcal{A}}_\alpha$	$-$	$-$	$+$	$-$
$\mathcal{P}$	$-$	$-$	$-$	$+$

Hence (P1–4) does not imply any of (PD), (PP), (PnF) or (PF). Hessenberg's Theorem says that (PP)$\Rightarrow$(PD), and recall that (PnF) and (PF) exclude each other. Apart from that we do not have any other implications: We have (PP)$\not\Rightarrow$(PnF) and (PP)$\not\Rightarrow$(PF), and (PnF)$\not\Rightarrow$(PD) and (PF)$\not\Rightarrow$(PD), and (PD,PnF)$\not\Rightarrow$(PP) and (PD,PF)$\not\Rightarrow$(PP).

6 Spaces

(6.1) Linear subspaces. Let $\mathcal{P}$ be a projective space fulfilling (P4), with points $\mathcal{V}$. A subset $\mathcal{U} \subseteq \mathcal{V}$ is called **linear**, if for all $P \neq Q \in \mathcal{U}$ we already have $PQ \subseteq \mathcal{U}$; we write $\mathcal{U} \leq \mathcal{V}$. Hence the set $\mathcal{U}$ of points, together with the lines $\{l \text{ line}; l \subseteq \mathcal{U}\}$, is a rank 2 subgeometry of $\mathcal{P}$, thus a projective space again, being called a **linear subspace** of $\mathcal{P}$; note that $\mathcal{U}$ might be degenerate due to failure of (P3), even if $\mathcal{P}$ is non-degenerate.

If $\mathcal{I}$ is a set and $\mathcal{U}_i \leq \mathcal{V}$ for all $i \in \mathcal{I}$, then $\bigcap_{i\in\mathcal{I}} \mathcal{U}_i \leq \mathcal{V}$ is linear again, the empty intersection being $\mathcal{V}$. Hence given a subset $\mathcal{S} \subseteq \mathcal{V}$, the set $\langle\mathcal{S}\rangle = \bigcap\{\mathcal{U} \leq \mathcal{V}; \mathcal{S} \subseteq$

$\mathcal{U}\} \leq \mathcal{V}$ is called the linear set **generated** by $\mathcal{S}$. Thus $\langle \mathcal{S} \rangle$ is the smallest linear subset containing $\mathcal{S}$, with respect to the partial order on all linear subsets given by set-theoretic inclusion. In particular, we have $\mathcal{S} = \langle \mathcal{S} \rangle$ if and only if $\mathcal{S} \leq \mathcal{V}$ is linear, and hence $\langle \langle \mathcal{S} \rangle \rangle = \langle \mathcal{S} \rangle$ for all $\mathcal{S} \subseteq \mathcal{V}$. For example, the empty set, any single point, any single line, and all of $\mathcal{V}$ are linear sets. Hence we have $\langle \emptyset \rangle = \emptyset$ and $\langle P \rangle = \{P\}$, as well as $\langle P, Q \rangle = PQ$ for all $P \neq Q \in \mathcal{V}$.

If there is a finite **generating set** $\mathcal{S}$ such that $\mathcal{V} = \langle \mathcal{S} \rangle$, then $\mathcal{P}$ is called **finitely generated**. In this case $\mathcal{P}$ has a minimal generating set, with respect to the partial order given by set-theoretic inclusion: To apply induction let $\mathcal{N} := \{n \in \mathbb{N}_0; \mathcal{P} \text{ has a generating set of cardinality } n\}$, then $\mathcal{N} \neq \emptyset$, thus $\mathcal{N}$ has a minimal element, where a generating set of minimal cardinality is minimal.

(6.2) Theorem. Let $\mathcal{P}$ be a projective space fulfilling (P4), with points $\mathcal{V}$, and let $\emptyset \neq \mathcal{U}, \mathcal{W} \leq \mathcal{V}$ such that $|\mathcal{U} \cup \mathcal{W}| \geq 2$. Then $\langle \mathcal{U}, \mathcal{W} \rangle = \bigcup_{\mathcal{U} \ni P \neq Q \in \mathcal{W}} PQ$.

In particular, for $Q \in \mathcal{V} \setminus \mathcal{U}$ we have $\langle \mathcal{U}, Q \rangle = \bigcup_{P \in \mathcal{U}} PQ$, and for all $Q' \in \langle \mathcal{U}, Q \rangle \setminus \mathcal{U}$ we have the **exchange property** $\langle \mathcal{U}, Q \rangle = \langle \mathcal{U}, Q' \rangle$.

Proof. Since $\langle \mathcal{U}, \mathcal{W} \rangle$ is linear, we conclude that $PQ \subseteq \langle \mathcal{U}, \mathcal{W} \rangle$ for all $P \in \mathcal{U}$ and $Q \in \mathcal{W}$ such that $P \neq Q$, hence we have $\mathcal{U} \cup \mathcal{W} \subseteq \mathcal{S} := \bigcup_{\mathcal{U} \ni P \neq Q \in \mathcal{W}} PQ \subseteq \langle \mathcal{U}, \mathcal{W} \rangle$. In particular, for $Q \in \mathcal{V} \setminus \mathcal{U}$ we have $\mathcal{U} \subseteq \mathcal{T} := \bigcup_{P \in \mathcal{U}} PQ \subseteq \langle \mathcal{U}, Q \rangle$.

To prove the converse we first show that $\mathcal{T}$ is linear, implying that $\langle \mathcal{U}, Q \rangle = \mathcal{T}$: Let $R \neq S \in \mathcal{T}$. Hence there are $R', S' \in \mathcal{U}$ such that $R \in R'Q$ and $S \in S'Q$; thus $RS \subseteq \langle R', S', Q \rangle$. If $Q \in RS$ then we may assume that $R \neq Q$, hence we have $RS = RQ = R'Q \subseteq \mathcal{T}$; hence we may assume that $Q \notin RS$. If $\{R, S\} \subseteq \mathcal{U}$ then we have $RS \subseteq \mathcal{U}$, hence we may assume that $R \notin \mathcal{U}$, and thus $R \neq R'$. If $R' = S'$ then we have $RS = RQ = R'Q = S'Q = SQ \subseteq \mathcal{T}$; thus we may assume that $R' \neq S'$. Then we have $RS \not\parallel R'S'$: If $S \neq S'$ this follows from $Q \in RR' \cap SS'$ using (P2'). Let now $X \in RS$, then we have $QX \cap RR' \ni Q \neq X \in QX \cap RS$, hence (P2") applied to the triangle with the sides $\{RR', RS, R'S'\}$ implies that $QX \not\parallel R'S' \subseteq \mathcal{U}$, hence $X \in \mathcal{T}$.

In particular, for $Q' \in \langle \mathcal{U}, Q \rangle \setminus \mathcal{U}$ we have $\langle \mathcal{U}, Q' \rangle \subseteq \langle \mathcal{U}, Q \rangle$, and there is $P \in \mathcal{U}$ such that $Q' \in PQ$, hence $Q \in PQ = PQ' \subseteq \langle \mathcal{U}, Q' \rangle$ implies $\langle \mathcal{U}, Q \rangle \subseteq \langle \mathcal{U}, Q' \rangle$.

We finally show that $\mathcal{S}$ is linear, implying that $\langle \mathcal{U}, \mathcal{W} \rangle = \mathcal{S}$: Let $R \neq R' \in \mathcal{S}$, hence there are $P, P' \in \mathcal{U}$ and $Q, Q' \in \mathcal{W}$, where $P \neq Q$ and $P' \neq Q'$, such that $R \in PQ$ and $R' \in P'Q'$; thus $RR' \subseteq \langle P, P', Q, Q' \rangle$. If $\{Q, Q'\} \subseteq \mathcal{U}$ then we have $RR' \subseteq \mathcal{U}$, hence we may assume that $Q \notin \mathcal{U}$. If $Q' \in \langle \mathcal{U}, Q \rangle$ then we have $RR' \subseteq \langle \mathcal{U}, Q \rangle \subseteq \mathcal{S}$, thus we may assume that $Q' \notin \langle \mathcal{U}, Q \rangle$, in particular $Q \neq Q'$. Let now $X \in RR' \subseteq \langle P, P', Q, Q' \rangle \subseteq \langle \langle \mathcal{U}, Q \rangle, Q' \rangle$. Hence there is $X' \in \langle \mathcal{U}, Q \rangle$ such that $X \in X'Q'$, and in turn there is $P'' \in \mathcal{U}$ such that $X' \in P''Q$. Hence we have $X \in \langle P'', Q, Q' \rangle \leq \langle \mathcal{W}, P'' \rangle$, where if $P'' \notin \mathcal{W}$ there is $Q'' \in \mathcal{W}$ such that $X \in P''Q'' \subseteq \mathcal{S}$. $\sharp$

(6.3) Bases. Let $\mathcal{P}$ be a projective space fulfilling (P4), with points $\mathcal{V}$. A subset $\mathcal{S} \subseteq \mathcal{V}$ is called **independent**, if for all $P \in \mathcal{S}$ we have $P \notin \langle \mathcal{S} \setminus \{P\} \rangle$; otherwise $\mathcal{S}$ is called **dependent**. Hence any subset of an independent set is independent again. For example, the empty set, any single point, and any pair of distinct points are independent sets; any 3 pairwise distinct points are independent if and only if they are non-collinear. An independent generating set of $\mathcal{V}$ is called a **basis** of $\mathcal{P}$.

A set $\mathcal{B} \subseteq \mathcal{V}$ is a basis if and only if $\mathcal{B}$ is a minimal generating set, with respect to the partial order given by set-theoretic inclusion; in particular, if $\mathcal{P}$ is finitely generated, then $\mathcal{P}$ has a finite basis: If $\mathcal{B}$ is a basis, assume that $\mathcal{B}$ is non-minimal, then there is $P \in \mathcal{B}$ such that $\langle \mathcal{B} \setminus \{P\} \rangle = \mathcal{V}$, hence $P \in \langle \mathcal{B} \setminus \{P\} \rangle$, a contradiction. If $\mathcal{B}$ is minimal, assume that there is $P \in \mathcal{B}$ such that $P \in \langle \mathcal{B} \setminus \{P\} \rangle$, hence $\mathcal{V} = \langle \mathcal{B} \rangle = \langle \mathcal{B} \setminus \{P\}, P \rangle = \langle \mathcal{B} \setminus \{P\} \rangle$, a contradiction.

A set $\mathcal{B} \subseteq \mathcal{V}$ is a basis if and only if $\mathcal{B}$ is a maximal independent set, with respect to the partial order given by set-theoretic inclusion: If $\mathcal{B}$ is a basis, then for any $P \in \mathcal{V} \setminus \mathcal{B}$ we have $P \in \langle \mathcal{B} \rangle$, hence $\mathcal{B} \,\dot{\cup}\, \{P\}$ is dependent, thus $\mathcal{B}$ is maximal independent. If $\mathcal{B}$ is maximal independent, then assume that there is $P \in \mathcal{V} \setminus \langle \mathcal{B} \rangle$, then the set $\mathcal{B} \,\dot{\cup}\, \{P\}$ is dependent, hence there is $Q \in \mathcal{B}$ such that $Q \in \langle \mathcal{B} \setminus \{Q\}, P \rangle$, thus since $Q \notin \langle \mathcal{B} \setminus \{Q\} \rangle$ the exchange property implies $P \in \langle \mathcal{B} \setminus \{Q\}, P \rangle = \langle \mathcal{B} \setminus \{Q\}, Q \rangle = \langle \mathcal{B} \rangle$, a contradiction.

(6.4) Proposition. Let $\mathcal{P}$ be a projective space fulfilling (P4), with points $\mathcal{V}$, let $\mathcal{S} \subseteq \mathcal{V}$ be finite and independent, and let $P \in \langle \mathcal{S} \rangle$. Then there is a unique minimal subset $\mathcal{T} \subseteq \mathcal{S}$, with respect to the partial order given by set-theoretic inclusion, such that $P \in \langle \mathcal{T} \rangle$, being called the **support** of P with respect to $\mathcal{S}$.

Proof. It suffices to show that for all subsets $\mathcal{S}', \mathcal{S}'' \subseteq \mathcal{S}$ we have $\langle \mathcal{S}' \cap \mathcal{S}'' \rangle = \langle \mathcal{S}' \rangle \cap \langle \mathcal{S}'' \rangle$, since then letting $\mathcal{T} := \bigcap \{\mathcal{S}' \subseteq \mathcal{S}; P \in \langle \mathcal{S}' \rangle\} \subseteq \mathcal{S}$ we have $P \in \bigcap \{\langle \mathcal{S}' \rangle; \mathcal{S}' \subseteq \mathcal{S}, P \in \langle \mathcal{S}' \rangle\} = \langle \mathcal{T} \rangle$: To this end, we have $\langle \mathcal{S}' \cap \mathcal{S}'' \rangle \subseteq \langle \mathcal{S}' \rangle \cap \langle \mathcal{S}'' \rangle$. Conversely, we proceed by induction on $|\mathcal{S}'|$, where the assertion is trivial for $\mathcal{S}' = \emptyset$. We may assume that $\mathcal{S}' \not\subseteq \mathcal{S}''$, hence let $Q \in \mathcal{S}' \setminus \mathcal{S}''$. Assume that there is $R \in (\langle \mathcal{S}' \rangle \cap \langle \mathcal{S}'' \rangle) \setminus \langle \mathcal{S}' \cap \mathcal{S}'' \rangle$.

Assume that $R \in \langle \mathcal{S}' \setminus \{Q\} \rangle$, hence by induction we have $R \in \langle \mathcal{S}' \setminus \{Q\} \rangle \cap \langle \mathcal{S}'' \rangle = \langle (\mathcal{S}' \setminus \{Q\}) \cap \mathcal{S}'' \rangle \leq \langle \mathcal{S}' \cap \mathcal{S}'' \rangle$, a contradiction. Thus $Q, R \in \langle \mathcal{S}' \rangle \setminus \langle \mathcal{S}' \setminus \{Q\} \rangle$, and the exchange property implies $Q \in \langle \mathcal{S}' \setminus \{Q\}, R \rangle$. Since $R \in \langle \mathcal{S}'' \rangle$ this yields $Q \in \langle \mathcal{S}' \setminus \{Q\}, \mathcal{S}'' \rangle$, since $Q \notin \mathcal{S}''$ contradicting the independence of $\mathcal{S}$. ♯

(6.5) Theorem: Steinitz's Base Change Theorem [1913]. Let $\mathcal{P}$ be a projective space fulfilling (P4), with points $\mathcal{V}$, having a finite basis $\mathcal{B}$. If $\mathcal{S} \subseteq \mathcal{V}$ is independent, then we have $|\mathcal{S}| \leq |\mathcal{B}|$, and there is $\mathcal{T} \subseteq \mathcal{B}$ such that $|\mathcal{S}| + |\mathcal{T}| = |\mathcal{B}|$ and $\mathcal{S} \,\dot{\cup}\, \mathcal{T}$ is a basis of $\mathcal{P}$.

Proof. We may assume that $\mathcal{S}$ is finite, and we proceed by induction on $|\mathcal{S}|$, where the assertion is trivial for $\mathcal{S} = \emptyset$. Hence let $Q \in \mathcal{S}$ and $\mathcal{S}' := \mathcal{S} \setminus \{Q\}$, and let $\mathcal{T}' \subseteq \mathcal{B}$ such that $\mathcal{S}' \dot\cup \mathcal{T}'$ is a basis of $\mathcal{P}$. Since $Q \in \langle \mathcal{S}', \mathcal{T}' \rangle$ let $\mathcal{M} \subseteq \mathcal{S}' \dot\cup \mathcal{T}'$ be the associated support of Q. Since $Q \notin \langle \mathcal{S}' \rangle$, there is $P \in \mathcal{M} \cap \mathcal{T}'$. Letting $\mathcal{T} := \mathcal{T}' \setminus \{P\}$ we show that $\mathcal{S} \dot\cup \mathcal{T} = (\mathcal{S}' \dot\cup \{Q\}) \dot\cup (\mathcal{T}' \setminus \{P\})$ is a basis of $\mathcal{P}$; note that if $Q \in \mathcal{T}'$ then $\mathcal{M} = \{Q\} = \{P\}$, thus in any case we have $\mathcal{S} \cap \mathcal{T} = \emptyset$:

Since $P \in \mathcal{M}$ but $P \notin \mathcal{S}' \dot\cup \mathcal{T}$, we conclude that $Q \notin \langle \mathcal{S}', \mathcal{T} \rangle$. Hence since $Q \in \langle \mathcal{S}', \mathcal{T}, P \rangle = \langle \mathcal{S}', \mathcal{T}' \rangle = \mathcal{V}$ the exchange property implies $\langle \mathcal{S}, \mathcal{T} \rangle = \langle \mathcal{S}', \mathcal{T}, Q \rangle = \langle \mathcal{S}', \mathcal{T}, P \rangle = \mathcal{V}$, thus $\mathcal{S} \dot\cup \mathcal{T}$ is a generating set.

Assume that there is $R \in \mathcal{S} \dot\cup \mathcal{T}$ such that $R \in \langle (\mathcal{S} \dot\cup \mathcal{T}) \setminus \{R\} \rangle$. If $R = Q$ then we have $Q \in \langle \mathcal{S}', \mathcal{T} \rangle$, a contradiction; if $R \in \mathcal{S}' \dot\cup \mathcal{T}$ then we have $R \in \langle (\mathcal{S}' \dot\cup \mathcal{T}') \setminus \{P, R\}, Q \rangle$, thus since $R \notin \langle (\mathcal{S}' \dot\cup \mathcal{T}') \setminus \{P, R\} \rangle$ the exchange property yields $Q \in \langle (\mathcal{S}' \dot\cup \mathcal{T}') \setminus \{P\} \rangle = \langle \mathcal{S}', \mathcal{T} \rangle$, a contradiction. $\sharp$

(6.6) Dimension. a) Let $\mathcal{P}$ be a finitely generated projective space fulfilling (P4), with points $\mathcal{V}$. Then any finite generating set contains a basis, and any independent set can be extended to a basis.

All bases of $\mathcal{P}$ are finite of the same cardinality $d + 1 \geq 0$, then $\dim(\mathcal{P}) := d \geq -1$ is called the **dimension** of $\mathcal{P}$; if $\mathcal{P}$ is not finitely generated we write $\dim(\mathcal{P}) := \infty$. For any subset $\mathcal{B} \subseteq \mathcal{V}$ the following are equivalent:

i) $\mathcal{B}$ is a basis;
ii) $\mathcal{B}$ is independent of maximal cardinality;
ii') $\mathcal{B}$ is independent such that $|\mathcal{B}| = \dim(\mathcal{P}) + 1$;
iii) $\mathcal{B}$ is a generating set of minimal cardinality;
iii') $\mathcal{B}$ is a generating set such that $|\mathcal{B}| = \dim(\mathcal{P}) + 1$.

b) We have $\dim(\mathcal{P}) \in \{-1, 0, 1\}$ if and only if $\mathcal{P} = \emptyset$, respectively $\mathcal{P}$ consists of a single point, respectively $\mathcal{P}$ consists of a single line. Moreover, $\mathcal{P}$ is non-degenerate if and only if $\dim(\mathcal{P}) \geq 2$.

The space $\mathcal{P}$ is a projective plane if and only if $\dim(\mathcal{P}) = 2$: Let $\mathcal{P}$ be a projective plane. Then $\mathcal{P}$ is generated by any non-collinear, that is independent, set $\{P, Q, R\}$, implying $\dim(\mathcal{P}) = 2$: Let $S \in \mathcal{V}$, then we may assume that $S \notin PQ$ and $R \neq S$, thus we have $RS \cap PQ \neq \emptyset$, hence $S \in \bigcup_{T \in PQ} RT = \langle P, Q, R \rangle$.

Conversely, let $\dim(\mathcal{P}) = 2$. If m is a line, then choosing points $P \neq Q \in m$ by Steinitz's Theorem there is a point R such that $\{P, Q, R\}$ is a basis. Then for any line $l \neq m = PQ$ we have $l \not\parallel m$, showing (P2): If $R \in l$ then let $R \neq S \in l$, hence there is $S' \in m$ such that $S \in RS'$, thus $l = RS = RS'$ and hence $S' \in l \cap m$. If $R \notin l$ then let $S \neq T \in l$ such that $S, T \notin m$, hence $\{R, S, T\}$ is non-collinear, and there are $S', T' \in m$ such that $S \in RS'$ and $T \in RT'$. Thus we have $SS' = RS \not\parallel RT = TT'$, hence (P2') implies that $l = ST \not\parallel S'T' = m$. $\sharp$

c) Let $\mathcal{U} \leq \mathcal{V}$ be linear. Then $\mathcal{U}$ is finitely generated such that $\dim(\mathcal{U}) \leq \dim(\mathcal{P})$, where equality holds if and only if $\mathcal{U} = \mathcal{V}$:

Any subset of $\mathcal{U}$ is independent in $\mathcal{U}$ if and only if it is independent in $\mathcal{V}$. Thus any independent subset of $\mathcal{U}$ has cardinality at most $\dim(\mathcal{P}) + 1$. Hence there is a maximal independent subset, thus a basis of $\mathcal{U}$, implying $\dim(\mathcal{U}) \leq \dim(\mathcal{P})$. If $\dim(\mathcal{U}) = \dim(\mathcal{P})$ then any basis $\mathcal{B}$ of $\mathcal{U}$ is a basis of $\mathcal{P}$, thus $\mathcal{U} = \langle \mathcal{B} \rangle = \mathcal{V}$. ♯

For $c \geq -1$ let $\mathcal{V}_c(\mathcal{P})$ be the set of c-dimensional linear subspaces of $\mathcal{P}$. Hence we have $\mathcal{V}_c(\mathcal{P}) = \emptyset$ for $c > d := \dim(\mathcal{P})$, as well as $\mathcal{V}_d(\mathcal{P}) = \{\mathcal{P}\}$ and $\mathcal{V}_{-1}(\mathcal{P}) = \{\emptyset\}$, while $\mathcal{V}_0(\mathcal{P}) = \mathcal{V}$ are the points, $\mathcal{V}_1(\mathcal{P})$ are the lines, $\mathcal{V}_2(\mathcal{P})$ are the planes, and $\mathcal{V}_{d-1}(\mathcal{P})$ are the **hyperplanes** of $\mathcal{P}$. Thus $\bigcup_{c=0}^{d-1} \mathcal{V}_c(\mathcal{P})$ is a geometry of rank d, the incidence relation given by set-theoretic inclusion, called the associated **projective geometry**, for $d \geq 2$ containing $\mathcal{P}$ as a subgeometry of rank 2.

(6.7) Theorem: Dimension formula. Let $\mathcal{P}$ be a projective space fulfilling (P4), with points $\mathcal{V}$, and let $\mathcal{U} \leq \mathcal{V}$ and $\mathcal{U}' \leq \mathcal{V}$ be finitely generated. Then we have $\dim(\mathcal{U}) + \dim(\mathcal{U}') = \dim(\langle \mathcal{U}, \mathcal{U}' \rangle) + \dim(\mathcal{U} \cap \mathcal{U}')$.

Proof. We may assume that $\mathcal{U} \neq \emptyset \neq \mathcal{U}'$, hence $d := \dim(\mathcal{U}) \geq 0$ and $d' := \dim(\mathcal{U}') \geq 0$. Since $\langle \mathcal{U}, \mathcal{U}' \rangle$ is finitely generated we have $e := \dim(\langle \mathcal{U}, \mathcal{U}' \rangle) \geq 0$. Let $c := \dim(\mathcal{U} \cap \mathcal{U}') \geq -1$. If $c = d$ then we have $\mathcal{U} = \mathcal{U} \cap \mathcal{U}'$, that is $\mathcal{U} \leq \mathcal{U}'$, implying $\langle \mathcal{U}, \mathcal{U}' \rangle = \mathcal{U}'$; hence we may assume that $d > c$ and $d' > c$. Let $\mathcal{C}$ be a basis of $\mathcal{U} \cap \mathcal{U}'$, and let $\mathcal{B} \subseteq \mathcal{U}$ and $\mathcal{B}' \subseteq \mathcal{U}'$ such that $\mathcal{C} \,\dot\cup\, \mathcal{B}$ and $\mathcal{C} \,\dot\cup\, \mathcal{B}'$ are bases of $\mathcal{U}$ and $\mathcal{U}'$, respectively; note that $|\mathcal{B}| = d - c \geq 1$ and $|\mathcal{B}'| = d' - c \geq 1$, while $|\mathcal{C}| = c + 1 \geq 0$.

Hence we have $\langle \mathcal{C} \cup \mathcal{B} \cup \mathcal{B}' \rangle = \langle \mathcal{U}, \mathcal{U}' \rangle$. Assume that $P \in \mathcal{U} \cap \langle \mathcal{B}' \rangle \subseteq \mathcal{U} \cap \mathcal{U}' = \langle \mathcal{C} \rangle$, hence considering the support of P with respect to $\mathcal{C} \,\dot\cup\, \mathcal{B}'$ shows that $P \in \langle \mathcal{B}' \rangle \cap \langle \mathcal{C} \rangle = \langle \mathcal{B}' \cap \mathcal{C} \rangle = \langle \emptyset \rangle = \emptyset$, a contradiction. Thus we have $\mathcal{U} \cap \langle \mathcal{B}' \rangle = \emptyset$, in particular $\mathcal{B} \cap \mathcal{B}' = \emptyset$. We show that $\mathcal{C} \,\dot\cup\, \mathcal{B} \,\dot\cup\, \mathcal{B}'$ is independent, implying that $e + 1 = |\mathcal{C} \,\dot\cup\, \mathcal{B} \,\dot\cup\, \mathcal{B}'| = (c + 1) + (d - c) + (d' - c) = d + d' - c + 1$:

Assume that $\mathcal{C} \,\dot\cup\, \mathcal{B} \,\dot\cup\, \mathcal{B}'$ is dependent, then we may assume that there is $P \in \mathcal{C} \,\dot\cup\, \mathcal{B}$ such that $P \in \langle (\mathcal{C} \,\dot\cup\, \mathcal{B}) \setminus \{P\}, \mathcal{B}' \rangle$. Assume that $(\mathcal{C} \,\dot\cup\, \mathcal{B}) \setminus \{P\} = \emptyset$, then we have $P \in \mathcal{U} \cap \langle \mathcal{B}' \rangle = \emptyset$, a contradiction. Hence we have $(\mathcal{C} \,\dot\cup\, \mathcal{B}) \setminus \{P\} \neq \emptyset$, and thus there are $Q \in \langle (\mathcal{C} \,\dot\cup\, \mathcal{B}) \setminus \{P\} \rangle \leq \mathcal{U}$ and $R \in \langle \mathcal{B}' \rangle$ such that $P \in QR$; note that necessarily $Q \neq R$. Assume that $|\mathcal{U} \cap QR| \geq 2$, then $R \in QR \subseteq \mathcal{U}$, a contradiction. Thus we have $P \in \mathcal{U} \cap QR = \{Q\}$, hence $P = Q \in \langle (\mathcal{C} \,\dot\cup\, \mathcal{B}) \setminus \{P\} \rangle$, contradicting the independence of $\mathcal{C} \,\dot\cup\, \mathcal{B}$. ♯

(6.8) Projective spaces over skew fields. a) Let $\mathbb{F}$ be a skew field, and let $V := \mathbb{F}^{(d+1) \times 1}$ where $d \geq 1$. We consider the projective space $\mathbb{P}^d(\mathbb{F})$, with points $\mathcal{V} := \{\langle v \rangle_{\mathbb{F}} \leq V; 0 \neq v \in V\}$, and proceed to show that $\dim(\mathbb{P}^d(\mathbb{F})) = d$:

Let $U \leq V$ be an $\mathbb{F}$-linear subspace with $\mathbb{F}$-basis $\mathcal{C} := \{u_0, \ldots, u_c\}$, where $c+1 = \dim_{\mathbb{F}}(U)$. Then we have $\mathcal{T} := \langle \langle u_0 \rangle_{\mathbb{F}}, \ldots, \langle u_c \rangle_{\mathbb{F}} \rangle = \{\langle u \rangle_{\mathbb{F}}; 0 \neq u \in U\} =: \mathcal{U} \subseteq \mathcal{V}$:

For $u, u' \in U$ we have $\langle u, u' \rangle_{\mathbb{F}} \leq U \leq V$, hence $\mathcal{U} \leq \mathcal{V}$ is linear, implying that $\mathcal{T} \subseteq \mathcal{U}$. Conversely, for $0 \neq u = \sum_{i=0}^{c} a_i u_i \in U$, where $0 \neq [a_0, \ldots, a_c] \in \mathbb{F}^{c+1}$, let $\mathcal{I}_u := \{i \in \{0, \ldots, c\}; a_i \neq 0\} \neq \emptyset$ be the **support** of u with respect to $\mathcal{C}$; note that $\mathcal{I}_u$ only depends on $\langle u \rangle_{\mathbb{F}}$. We proceed by induction on $|\mathcal{I}_u|$, where for $\mathcal{I}_u = \{i\}$ we have $\langle u \rangle_{\mathbb{F}} = \langle u_i \rangle_{\mathbb{F}} \in \mathcal{T}$. Thus let $|\mathcal{I}_u| \geq 2$, choose $i \in \mathcal{I}_u$ and let

$v := u - a_i u_i \neq 0$, hence we have $\mathcal{I}_v := \mathcal{I}_u \setminus \{i\} \neq \emptyset$. Then by induction we have $\langle u\rangle_{\mathbb{F}} = \langle v + a_i u_i\rangle_{\mathbb{F}} \leq \langle v, u_i\rangle_{\mathbb{F}} \in \langle \mathcal{T}, \langle u_i\rangle_{\mathbb{F}}\rangle = \mathcal{T}$, showing that $\mathcal{U} \subseteq \mathcal{T}$. ♯

Since for any $i \in \{0, \ldots, c\}$ we have $U_i := \langle u_j; i \neq j \in \{0, \ldots, c\}\rangle_{\mathbb{F}} < U$, and thus $\langle\langle u_j\rangle_{\mathbb{F}}; i \neq j \in \{0, \ldots, c\}\rangle = \{\langle u\rangle_{\mathbb{F}}; 0 \neq u \in U_i\} \subset \mathcal{U}$, we conclude that $\{\langle u_0\rangle_{\mathbb{F}}, \ldots, \langle u_c\rangle_{\mathbb{F}}\} \subseteq \mathcal{U}$ is a minimal generating set, hence $\{\langle u_0\rangle_{\mathbb{F}}, \ldots, \langle u_c\rangle_{\mathbb{F}}\}$ is independent, and we have $\dim(\mathcal{U}) = c = \dim_{\mathbb{F}}(U) - 1$; in particular $\dim(\mathcal{V}) = d$.

b) We describe the linear subspaces of $\mathbb{P}(V) := \mathbb{P}^d(\mathbb{F})$: If $U \leq V$ then we have already shown that $\mathcal{U} := \{\langle u\rangle_{\mathbb{F}}; 0 \neq u \in U\} \leq \mathcal{V}$ is linear, hence $\mathbb{P}(U)$, with points $\mathcal{U}$, is a linear subspace. Conversely, if $\mathcal{U} \leq \mathcal{V}$ is linear with basis $\{\langle u_0\rangle_{\mathbb{F}}, \ldots, \langle u_c\rangle_{\mathbb{F}}\}$, where $c = \dim(\mathcal{U})$, then letting $U := \langle u_0, \ldots, u_c\rangle_{\mathbb{F}} \leq V$ and choosing an $\mathbb{F}$-basis $\mathcal{C} \subseteq \{u_0, \ldots, u_c\}$ of U yields $\mathcal{U} = \langle\langle u_0\rangle_{\mathbb{F}}, \ldots, \langle u_c\rangle_{\mathbb{F}}\rangle \leq \mathbb{P}(U) = \{\langle u\rangle_{\mathbb{F}}; 0 \neq u \in U\} = \langle\langle u\rangle_{\mathbb{F}}; u \in \mathcal{C}\rangle \leq \langle\langle u_0\rangle_{\mathbb{F}}, \ldots, \langle u_c\rangle_{\mathbb{F}}\rangle = \mathcal{U}$; since $\dim_{\mathbb{F}}(U) = \dim(\mathcal{U}) + 1$ we conclude that $\{u_0, \ldots, u_c\}$ is $\mathbb{F}$-linearly independent.

Hence, if $\langle v_0\rangle_{\mathbb{F}}, \ldots, \langle v_c\rangle_{\mathbb{F}} \in \mathcal{V}$ are pairwise distinct, where $c \in \mathbb{N}_0$, then we conclude that $\{\langle v_0\rangle_{\mathbb{F}}, \ldots, \langle v_c\rangle_{\mathbb{F}}\} \subseteq \mathcal{V}$ is independent if and only if $\{v_0, \ldots, v_c\} \subseteq V$ is $\mathbb{F}$-linearly independent. Thus, letting $v_j = [x_{j0}, \ldots, x_{jd}]^{\mathrm{tr}} \in \mathbb{F}^{(d+1)\times 1} = V$ be homogeneous coordinates, then $\{\langle v_0\rangle_{\mathbb{F}}, \ldots, \langle v_c\rangle_{\mathbb{F}}\} \subseteq \mathcal{V}$ is independent if and only if the matrix $B := [x_{ji}]_{ij} \in \mathbb{F}^{(d+1)\times(c+1)}$ has full column rank $\mathrm{rk}(B) = c+1$.

Hence if $\{v_0, \ldots, v_c\} \subseteq V$ is $\mathbb{F}$-linearly independent, the homogeneous coordinates of the points in the c-dimensional linear subspace $\mathbb{P}(\langle v_0, \ldots, v_c\rangle_{\mathbb{F}})$ are given by the $\mathbb{F}$-subspace of solutions of the system of linear equations $A \cdot [x_0, \ldots, x_d]^{\mathrm{tr}} = 0 \in \mathbb{F}^{(d-c)\times 1}$, where $A \in \mathbb{F}^{(d-c)\times(d+1)}$ has full row rank $\mathrm{rk}(A) = d - c$ fulfilling $A \cdot B = 0 \in \mathbb{F}^{(d-c)\times(c+1)}$. In particular, a hyperplane is described by the $\mathbb{F}$-subspace of solutions of a single linear equation.

c) For $c \geq 0$ let $\mathcal{V}_c(V)$ be the set of c-dimensional $\mathbb{F}$-linear subspaces of V. Hence we have $\mathcal{V}_c(V) = \emptyset$ for $c > d + 1$ and $\mathcal{V}_{d+1}(V) = \{V\}$ and $\mathcal{V}_0(V) = \{\{0\}\}$. Thus $\bigcup_{c=1}^{d} \mathcal{V}_c(V)$ is a geometry of rank d, the incidence relation given by set-theoretic inclusion, called the **projective geometry** associated with V. Hence $\alpha\colon \bigcup_{c=1}^{d} \mathcal{V}_c(V) \to \bigcup_{c=0}^{d-1} \mathcal{V}_c(\mathbb{P}(V))\colon U \mapsto \mathcal{U} := \{\langle u\rangle_{\mathbb{F}}; 0 \neq u \in U\}$ is an isomorphism between the projective geometries associated with V and with $\mathbb{P}(V)$, respectively; note that $\alpha(\{0\}) = \emptyset$ and $\alpha(V) = \mathcal{V}$ is well-defined as well.

We have $\alpha(U \cap U') = \alpha(U) \cap \alpha(U')$, for all $U, U' \leq V$. The dimension formulae for vector spaces and for linear subspaces, respectively, yield $\dim_{\mathbb{F}}(U + U') = \dim_{\mathbb{F}}(U) + \dim_{\mathbb{F}}(U') - \dim_{\mathbb{F}}(U \cap U') = \dim(\alpha(U)) + \dim(\alpha(U')) - \dim(\alpha(U) \cap \alpha(U')) + 1 = \dim(\langle \alpha(U), \alpha(U')\rangle) + 1$, which since $\langle \alpha(U), \alpha(U')\rangle \leq \alpha(U + U')$ implies $\alpha(U + U') = \langle \alpha(U), \alpha(U')\rangle$ as well.

(6.9) Theorem. Let $\mathcal{P}$ be a projective space of dimension $\dim(\mathcal{P}) \geq 3$. Then $\mathcal{P}$ is Desarguesian.

Proof. Omitted. ♯

(6.10) Theorem. Let $\mathcal{P}$ be a finitely generated Desarguesian space of dimension $\dim(\mathcal{P}) \geq 2$. Then there is a skew field $\mathbb{F}$ such that $\mathcal{P} \cong \mathbb{P}^d(\mathbb{F})$.

Proof. Omitted. $\sharp$

(6.11) Corollary. Let $\mathcal{P}$ be a projective plane. Then $\mathcal{P}$ is Desarguesian if and only if $\mathcal{P}$ is a proper linear subspace of a projective space.

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7 Exercises (in German)

(7.1) Aufgabe: Ordnungsrelationen.
a) Es seien $\leq$ eine partielle Ordnung auf der Menge M, und $N \subseteq M$. Man gebe Definitionen der Begriffe **kleinste obere Schranke** und **größte untere Schranke**, sowie der **maximalen** und **minimalen** Elemente von N an.
b) Für eine partielle Ordnung $\leq$ auf der Menge M schreibt man $x < y$, falls $x \leq y$ und $x \neq y$ gilt. Man zeige: Gilt $x \leq y$ oder $y \leq x$ für alle $x, y \in M$, so gilt für alle $x, y \in M$ genau eine der Beziehungen $x < y$ oder $x = y$ oder $y < x$. Solche Relationen werden als **Ordnungen** bezeichnet.
c) Man zeige: Die Potenzmenge einer Menge M ist durch $\subseteq$ partiell geordnet. Die Menge $\mathbb{N} := \{1, 2, \ldots\}$ ist sowohl durch $\{[x, y] \in \mathbb{N}^2; x \mid y\}$ als auch durch $\leq$ partiell geordnet. Sind dies Ordnungen? Was gilt für $\geq$ auf $\mathbb{N}$?

(7.2) Aufgabe: Äquivalenzrelationen.
a) Es sei $\sim$ eine Äquivalenzrelation auf der Menge M, und für $x \in M$ sei $[x]_\sim := \{y \in M; x \sim y\} \subseteq M$ die zugehörige **Äquivalenzklasse**. Man zeige: M ist die disjunkte Vereinigung der verschiedenen $\sim$-Äquivalenzklassen.
b) Man zeige umgekehrt: Ist eine Menge M die disjunkte Vereinigung von Teilmengen $M_i \subseteq M$, wobei $i \in \mathcal{I}$ für eine Indexmenge $\mathcal{I}$, so kann man die M_i als die Äquivalenzklassen einer geeigneten Äquivalenzrelation auffassen.
c) Es sei R eine symmetrische, transitive Relation auf der Menge M mit folgender Zusatzeigenschaft: Für jedes $x \in M$ gibt es ein $y \in M$ mit $[x, y] \in R$. Man zeige: R ist eine Äquivalenzrelation. Kann man auf die Voraussetzung der Zusatzeigenschaft verzichten?
d) Man zeige: Die Relation $\{[x, x] \in M^2; x \in M\}$ ist eine Äquivalenzrelation. Man gebe die Äquivalenzklassen an.
e) Man untersuche die Relationen $R := \{[x, y] \in \mathbb{Z}^2; x \mid y\}$ und $R_n := \{[x, y] \in \mathbb{Z}^2; x \equiv y \pmod{n}\}$, wobei $n \in \mathbb{N}$, auf Reflexivität, Symmetrie und Transitivität. Sind dies Äquivalenzrelationen? Gegebenenfalls gebe man die Äquivalenzklassen an.

(7.3) Aufgabe: Weihnachtsbäume.
a) Ein König befiehlt seinem Gärtner, zur Weihnachtszeit im Schloßpark fünf gerade Reihen mit je vier Weihnachtsbäumen zu pflanzen. Daraufhin bestellt der Gärtner umgehend zwanzig Tannenbäume. Dann erfährt der Gärtner, daß bis Heiligabend nur zehn Bäume geliefert werden können. Kann er die zehn Bäume so pflanzen, daß die Bedingung trotzdem erfüllt wird?
b)* Nun hat der König noch eine weitere Aufgabe für den Gärtner: Er gibt ihm zehn Silbertafeln, auf denen die Zahlen von 1 bis 10 eingraviert sind. Dann befiehlt er ihm, die Weihnachtsbäume so mit den Tafeln zu schmücken, daß die Summe der Zahlen auf den Tafeln an den Bäumen in einer Reihe für alle fünf Reihen die gleiche ist. Kann der Gärtner die Weihnachtsbäume so schmücken?

(7.4) Aufgabe: Parkette.
Ein **Polygon** ist eine kompakte Teilmenge der Euklidischen Ebene, deren Rand eine geschlossene, einfache, aus endlich vielen geradlinigen Stücken bestehende Kurve ist. Ein **Parkett** ist eine polygonale Überdeckung der Euklidischen Ebene, so daß jeder Punkt im Inneren höchstens eines Polygons liegt.
a) Man zeige: Die Menge $\mathcal{T}$ aller Ecken, Kanten und Polygone eines Parketts ist zusammen mit der mengentheoretischen Inklusion eine Geometrie. Hat $\mathcal{T}$ einen Rang? Wenn ja, welchen?
b) Man definiert eine andere Inzidenzrelation wie folgt: Zwei Elemente von $\mathcal{T}$ seien inzident, wenn sie nicht-leeren Schnitt haben. Man zeige: Die Menge $\mathcal{T}$ ist auch eine Geometrie bezüglich dieser Inzidenzrelation Hat $\mathcal{T}$ bezüglich dieser Inzidenzrelation einen Rang? Wenn ja, welchen?
c)* Ein Parkett heißt **regulär**, wenn es ein $n \in \mathbb{N}$ gibt, so daß jedes Polygon ein reguläres n-Eck ist. Man bestimme alle regulären Parkette.

(7.5) Aufgabe: Würfel und Tetraeder.
a) Es seien $\mathcal{C}$ und $\mathcal{T}$ die aus den Ecken, Kanten und Flächen des Euklidischen Würfels bzw. Tetraeders gebildeten Teilgeometrien vom Rang 3. Wieviele Fahnen besitzen sie jeweils? Wieviele davon sind maximal?
b) Man betrachte die aus den Ecken und Kanten, den Ecken und Flächen bzw. den Kanten und Flächen gebildeten Teilgeometrien von $\mathcal{C}$ und $\mathcal{T}$ vom Rang 2. Welche der Axiome für affine bzw. projektive Ebenen sind jeweils erfüllt?

(7.6) Aufgabe: Paarmengen.
Für $n \geq 2$ seien $\mathcal{X}_n := \{1, \ldots, n\}$, und $\mathcal{Y}_n$ die Menge der zwei-elementigen Teilmengen von $\mathcal{X}_n$. Weiter seien $\mathcal{P}_n$ und $\mathcal{P}_n^*$ die Geometrien vom Rang 2 mit Punkt- und Geradenmenge $\mathcal{X}_n$ und $\mathcal{Y}_n$ bzw. $\mathcal{Y}_n$ und $\mathcal{X}_n$, wobei die Inzidenzrelation jeweils durch mengentheoretische Inklusion gegeben sei.

Wieviele Punkte und Geraden haben diese Geometrien jeweils? Wieviele Punkte inzidieren jeweils mit einer Geraden, wieviele Geraden jeweils mit einem Punkt? Welche der Axiome für affine bzw. projektive Ebenen sind erfüllt? Sind es Teilgeometrien der Euklidischen Ebene?

(7.7) Aufgabe: Euler-Quadrate.
a) Ein **lateinisches Quadrat** der Ordnung $q \in \mathbb{N}$ ist eine Matrix $A \in \mathbb{Z}^{q \times q}$, so daß in jeder Zeile und jeder Spalte von A jede der Zahlen $\{1, \ldots, q\}$ vorkommt. Lateinische Quadrate $A = [a_{ij}]_{ij}$ und $B = [b_{ij}]_{ij}$ der Ordnung q heißen **orthogonal**, wenn die q^2 Paare $[a_{ij}, b_{ij}]$ für $i, j \in \{1, \ldots, q\}$ paarweise verschieden sind. Ein **Euler-** oder **griechisch-lateinisches Quadrat** ist ein Paar orthogonaler lateinischer Quadrate.

Man zeige: Ist $q \geq 3$ eine Primzahlpotenz, so gibt es ein Euler-Quadrat der Ordnung q. Was passiert im Fall $q = 2$?

b) Was haben Euler-Quadrate mit dem folgenden Problem zu tun?

Beim Divisionsball ordnet jedes von 6 Regimentern für jeden der 6 Dienstgrade je einen Offizier für eine besondere Aufgabe ab: Die 36 Offiziere sollen zur Feier des Tages so im Quadrat aufgestellt werden, daß in jeder Zeile und jeder Spalte genau ein Offizier jeden Regiments und jeden Dienstgrades steht.

(7.8) Aufgabe: Isomorphismen.
Es seien $\mathcal{G}$ und $\mathcal{G}'$ Geometrien vom Rang 2 mit Punkten $\mathcal{V}$ bzw. $\mathcal{V}'$, für die das Axiom (A1) gilt und jede Gerade mindestens 2 Punkte enthält.
a) Man zeige: Ein Isomorphismus $\mathcal{G} \to \mathcal{G}'$ ist eindeutig durch die von ihm induzierte Bijektion $\mathcal{V} \to \mathcal{V}'$ bestimmt.
a) Man zeige: Ist $\alpha\colon \mathcal{V} \to \mathcal{V}'$ eine Bijektion, so daß für alle $P, Q, R \in \mathcal{V}$ die Menge $\{P, Q, R\} \subseteq \mathcal{V}$ genau dann kollinear ist, wenn $\{\alpha(P), \alpha(Q), \alpha(R)\} \subseteq \mathcal{V}'$ kollinear ist, so kann α zu einem Isomorphismus $\mathcal{G} \to \mathcal{G}'$ fortgesetzt werden.

(7.9) Aufgabe: Degenerierte affine Ebenen.
Man gebe alle Geometrien vom Rang 2 an, die die Axiome (A1) und (A2), aber nicht das Axiom (A3) erfüllen.

(7.10) Aufgabe: Degenerierte projektive Ebenen.
Man gebe alle Geometrien vom Rang 2 an, die die Axiome (P1), (P2) und (P3), aber nicht das Axiom (P4) erfüllen.

(7.11) Aufgabe: Axiome für projektive Ebenen.
Man zeige, daß keine drei der Axiome (P1–4) das jeweils vierte implizieren.

(7.12) Aufgabe: Vierecke in projektiven Ebenen.
a) Man zeige: Jede projektive Ebene besitzt ein Viereck.
b) Man zeige: Erfüllt eine Geometrie $\mathcal{G}$ vom Rang 2 die Axiome (P1) und (P2) und besitzt sie ein Viereck, so ist $\mathcal{G}$ eine projektive Ebene.

(7.13) Aufgabe: Homogenes Modell projektiver Ebenen.
Es seien $\mathbb{A}^3(\mathbb{F})$ der affine Raum vom Rang 3 über dem Körper $\mathbb{F}$, und $\mathbb{P}^2(\mathbb{F})$ die Teilgeometrie vom Rang 2 von $\mathbb{A}^3(\mathbb{F})$, deren Punkte und Geraden die affinen Geraden bzw. Flächen sind, die den Nullpunkt enthalten. Man zeige explizit: $\mathbb{P}^2(\mathbb{F})$ ist eine projektive Ebene.

(7.14) Aufgabe: Homogene Koordinaten.
Im homogenen Modell der reellen projektive Ebene beschreibe man die Lage der folgenden Objekte im Euklidischen Raum, und berechne jeweils
a) die Gleichungen der Geraden durch die Punkte
i) $[1, 0, 1]^{\mathrm{tr}}$ und $[1, 2, 3]^{\mathrm{tr}}$, **ii)** $[1, 1, 0]^{\mathrm{tr}}$ und $[2, 3, 1]^{\mathrm{tr}}$, **iii)** $[1, 1, 0]^{\mathrm{tr}}$ und $[2, 3, 0]^{\mathrm{tr}}$,
b) die Schnittpunkte der Geraden gegeben durch die Gleichungen
i) $x + z = 0$ und $x + 2y + 3z = 0$, **ii)** $x + y = 0$ und $2x + 3y + z = 0$,
iii) $x + y = 0$ und $2x + 3y = 0$.

(7.15) Aufgabe: Automorphismen von $\mathbb{A}^2(2)$ und $\mathbb{P}^2(2)$.
a) Man bestimme die Automorphismengruppe der affinen Ebene $\mathbb{A}^2(2)$ als Gruppe von Permutationen ihrer Punkte.
b) Wieviele Elemente hat die Automorphismengruppe der projektiven Ebene $\mathbb{P}^2(2)$? Welche von ihnen sind Fortsetzungen von Automorphismen von $\mathbb{A}^2(2)$?

(7.16) Aufgabe: Endliche Ebenen.
Man konstruiere affine Ebenen der Ordnung 3 bzw. 4 sowie ihre projektiven Vervollständigungen.

(7.17) Aufgabe: Satz von Bruck-Ryser.
Man zeige: Es gibt keine projektive Ebene der Ordnung $q \equiv 6 \pmod 8$.

(7.18) Aufgabe: Axiome für endliche Ebenen.
Es sei $\mathcal{G}$ eine Geometrie vom Rang 2, die das Axiom (A1) erfüllt. Man zeige:
a) Gibt es $q \geq 2$, so daß $\mathcal{G}$ genau q^2 Punkte besitzt und auf jeder Geraden genau q Punkte liegen, so ist $\mathcal{G}$ eine affine Ebene.
b) Gibt es $q \geq 2$, so daß $\mathcal{G}$ genau q^2+q+1 Punkte besitzt und auf jeder Geraden genau $q+1$ Punkte liegen, so ist $\mathcal{G}$ eine projektive Ebene.

(7.19) Aufgabe: Inzidenzmatrizen.
a) Die **Inzidenzmatrix** $A = [a_{ij}]_{ij} \in \mathbb{Z}^{v \times v}$ einer projektiven Ebene der Ordnung $q \geq 2$ mit Punkten $\{P_1, \ldots, P_v\}$ und Geraden $\{l_1, \ldots, l_v\}$, wobei $v = q^2 + q + 1$, ist definiert durch $a_{ij} := 1$, falls $P_i \in l_j$, und $a_{ij} := 0$, falls $P_i \notin l_j$.

Man zeige: Es gilt $AA^{\text{tr}} = A^{\text{tr}}A = qE_v + J_v$, wobei $E_v \in \mathbb{Z}^{v \times v}$ die Einheitsmatrix und $J_v := [1]_{ij} \in \mathbb{Z}^{v \times v}$ die **Alles-1-Matrix** sind.

b) Umgekehrt zeige man: Sind $q \geq 2$ und $v := q^2+q+1$, sowie $A = [a_{ij}]_{ij} \in \mathbb{Z}^{v \times v}$ mit $a_{ij} \geq 0$ und $AA^{\text{tr}} = A^{\text{tr}}A = qE_v + J_v$, so ist A die Inzidenzmatrix einer projektiven Ebene der Ordnung q.

(7.20) Aufgabe: Zassenhaus-Algorithmus.
a) Es seien $\mathbb{F}$ ein Schiefkörper und $U, V \leq \mathbb{F}^n$, wobei $n \in \mathbb{N}$. Wie kann man durch Anwendung des Gauß-Algorithmus auf die Matrix $\left[\begin{array}{c|c} B & B \\ \hline C & 0 \end{array}\right] \in \mathbb{F}^{(m+l) \times 2n}$, wobei $B \in \mathbb{F}^{m \times n}$ und $C \in \mathbb{F}^{l \times n}$ Matrizen mit Zeilenraum U bzw. V sind, gleichzeitig $\mathbb{F}$-Basen von $U + V \leq \mathbb{F}^n$ und $U \cap V \leq \mathbb{F}^n$ berechnen?
b) Man berechne $\mathbb{Q}$-Basen von $U + V \leq \mathbb{Q}^4$ und $U \cap V \leq \mathbb{Q}^4$, wobei $U := \langle [3, -7, 2, -4], [2, -1, 5, 1] \rangle_{\mathbb{Q}} \leq \mathbb{Q}^4$ und $V := \langle [5, 7, 1, 6], [1, 7, -4, 4] \rangle_{\mathbb{Q}} \leq \mathbb{Q}^4$.

(7.21) Aufgabe: Schiefkörper.
Es seien $\mathbb{F}$ ein Schiefkörper sowie $a \in \mathbb{F}$ und $0 \neq b \in \mathbb{F}$ mit $a \neq b^{-1}$. Man zeige die **Hua-Identität**: Es gilt $((a - b^{-1})^{-1} - a^{-1})^{-1} = aba - a$.

(7.22) Aufgabe: Hamilton-Quaternionen.
Die Menge der **Hamilton-Quaternionen** sei definiert als

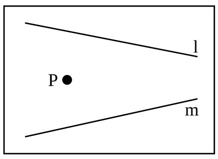
$$\mathbb{H} := \left\{ \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \in \mathbb{C}^{2\times 2}; \alpha, \beta \in \mathbb{C} \right\}.$$

a) Man zeige: $\mathbb{H}$ ist ein $\mathbb{C}$-Vektorraum mit $\dim_{\mathbb{C}}(\mathbb{H}) = 2$ und ein $\mathbb{R}$-Vektorraum mit $\dim_{\mathbb{R}}(\mathbb{H}) = 4$. Man gebe eine $\mathbb{C}$-Basis und eine $\mathbb{R}$-Basis von $\mathbb{H}$ an.
b) Man zeige: $\mathbb{H}$ ist ein nicht-kommutativer Schiefkörper. Man bestimme das **Zentrum** $Z(\mathbb{H}) := \{q \in \mathbb{H}; pq = qp$ für alle $p \in \mathbb{H}\}$. Wie kann man $\mathbb{R}$ und $\mathbb{C}$ als Teilkörper von $\mathbb{H}$ auffassen?

(7.23) Aufgabe: Pappussche Räume.
Man betrachte die Pappus-Konfiguration, siehe Tabelle 6, in einem Pappusschen Raum, und es sei $S \in l \cap l'$. Liegt der Punkt S notwendig auf der Geraden AB?

(7.24) Aufgabe: Desarguessche Räume.
Man betrachte die Desargues-Konfiguration, siehe Tabelle 7, in einem Desarguesschen Raum. Man zeige, daß jeder Punkt die Rolle des Zentrums S, und daß jede Gerade die Rolle der Achse $AB = BC = AC$ übernehmen kann.

(7.25) Aufgabe: Desarguessche Ebenen.
In der reellen affinen Ebene seien Geraden $l \not\parallel m$ gegeben, die sich 'außerhalb des Papiers' schneiden. Außerdem sei P ein Punkt 'auf dem Papier', also $P \notin l \cap m$. Man konstruiere 'auf dem Papier' die Gerade durch P und $l \cap m$.



(7.26) Aufgabe: Moulton-Ebenen.
Es sei $\widehat{\mathcal{A}}_\alpha$ die Moulton-Ebene mit Parameter $\alpha > 0$.
a) Man zeige explizit, daß $\widehat{\mathcal{A}}_\alpha$ genau für $\alpha = 1$ Pappussch ist.
b) Man zeige, daß $\widehat{\mathcal{A}}_\alpha$ nicht-Fanosch ist.

(7.27) Aufgabe: Lineare Teilräume.
Es seien $\mathcal{P}$ ein endlich erzeugter projektiver Raum, der (P4) erfüllt, mit Punktmenge $\mathcal{V}$, und $\mathcal{U}, \mathcal{W} \leq \mathcal{V}$ lineare Teilräume. Man zeige: Die Vereinigung $\mathcal{U} \cup \mathcal{W} \subseteq \mathcal{V}$ ist genau dann linear, wenn $\mathcal{U} \subseteq \mathcal{W}$ oder $\mathcal{W} \subseteq \mathcal{U}$ gilt.

(7.28) Aufgabe: Hyperebenen.
Es sei $\mathcal{P}$ ein endlich erzeugter projektiver Raum, der (P4) erfüllt. Man zeige:
a) Sind $\mathcal{H}_1, \ldots, \mathcal{H}_k$ Hyperebenen von $\mathcal{P}$, so gilt $\dim(\bigcap_{i=1}^k \mathcal{H}_i) \geq \dim(\mathcal{P}) - k$.

b) Jeder lineare Teilraum von $\mathcal{P}$ kann als Schnitt von Hyperebenen geschrieben werden. Wieviele Hyperebenen braucht man dazu mindestens?

(7.29) Aufgabe: Windschiefe Teilräume.
Es sei $\mathcal{P}$ ein endlich erzeugter projektiver Raum, der (P4) erfüllt. Eine Menge $\mathcal{M}$ paarweise disjunkter linearer Teilräume heißt **windschief**, und eine Gerade l heißt eine **Transversale** für $\mathcal{M}$, falls $|l \cap \mathcal{U}| = 1$ für alle $\mathcal{U} \in \mathcal{M}$ gilt.
a) Es seien $l \neq m$ parallele Geraden, und P ein Punkt mit $P \notin l$ und $P \notin m$. Man zeige: Es gibt höchstens eine Transversale für $\{l, m\}$, die P enthält; ist $\dim(\mathcal{P}) = 3$, so gibt es eine solche Transversale. Was passiert für $\dim(\mathcal{P}) \neq 3$?
b) Es seien $\mathcal{U}$ und $\mathcal{W}$ windschiefe Teilräume mit $\dim(\mathcal{U}) = \dim(\mathcal{W}) = \frac{\dim(\mathcal{P})-1}{2}$, und P ein Punkt mit $P \notin \mathcal{U}$ und $P \notin \mathcal{W}$. Man zeige: Es gibt genau eine Transversale für $\{\mathcal{U}, \mathcal{W}\}$, die P enthält.

(7.30) Aufgabe: Satz von Gallucci.
Es seien $\mathbb{F}$ ein Schiefkörper, und $\mathcal{L} := \{l_1, l_2, l_3\}$ und $\mathcal{M} := \{m_1, m_2, m_3\}$ jeweils paarweise verschiedene windschiefe Geraden in $\mathbb{P}^3(\mathbb{F})$, so daß $\mathcal{L}$ aus Transversalen für $\mathcal{M}$, und $\mathcal{M}$ aus Transversalen für $\mathcal{L}$ besteht. Man zeige: Genau dann gilt $l \not\parallel m$, für jede Transversale $l \notin \mathcal{L}$ für $\mathcal{M}$ und jede Transversale $m \notin \mathcal{M}$ für $\mathcal{L}$, wenn $\mathbb{F}$ ein Körper ist.

(7.31) Aufgabe: Punkte in allgemeiner Lage.
Es seien $\mathbb{F}$ ein Körper und $V := \mathbb{F}^{(d+1)\times 1}$, für $d \geq 1$. Eine Menge $\mathcal{S}$ von Punkten von $\mathbb{P}^d(\mathbb{F})$ heißt **in allgemeiner Lage**, wenn $|\mathcal{S}| \geq d+1$ und jede $(d+1)$-elementige Teilmenge von $\mathcal{S}$ eine Basis ist. Man zeige: Ist $|\mathbb{F}| \geq d$, so ist die Menge $\{\langle [1, t, t^2, \ldots, t^d]^{\mathrm{tr}} \rangle_{\mathbb{F}} \leq V; t \in \mathbb{F}\} \,\dot\cup\, \{\langle [0, \ldots, 0, 1]^{\mathrm{tr}} \rangle_{\mathbb{F}} \leq V\}$ in allgemeiner Lage.

(7.32) Aufgabe: Dualität.
Es seien $\mathbb{F}$ ein Körper und $d \geq 2$, und es sei $\mathcal{G}$ die Teilgeometrie vom Rang 2 der projektiven Geometrie von $\mathbb{P}^d(\mathbb{F})$, die aus den Punkten und den Hyperebenen von $\mathbb{P}^d(\mathbb{F})$ besteht. Man zeige: Es gilt $\mathcal{G} \cong \mathcal{G}^*$. Welche bekannte Aussage erhält man für den Fall $d = 2$?

(7.33) Aufgabe: Endliche Räume.
Es sei $\mathcal{P}$ ein endlich erzeugter nicht-ausgearteter projektiver Raum. Man zeige die Äquivalenz der folgenden Aussagen: **i)** $\mathcal{P}$ ist endlich; **ii)** $\mathcal{P}$ besitzt nur endlich viele Punkte; **iii)** $\mathcal{P}$ besitzt nur endlich viele Geraden; **iv)** $\mathcal{P}$ hat eine Gerade, die nur endlich viele Punkte besitzt. Was passiert im Fall $\dim(\mathcal{P}) \leq 1$?

(7.34) Aufgabe: Teilräume endlicher Räume.
Es seien $\mathcal{P}$ ein endlicher projektiver Raum, der (P4) erfüllt, der Dimension $d := \dim(\mathcal{P}) \geq 1$, und $\mathcal{V}_c(\mathcal{P})$ die Menge der c-dimensionalen linearen Teilräume

von $\mathcal{P}$, für $c \in \{0, \ldots, d\}$. Es sei weiter $q \geq 2$ die Ordnung von $\mathcal{P}$, und für $i \in \mathbb{N}_0$ sei $[i]_q := \frac{q^{i+1}-1}{q-1} = \sum_{j=0}^{i} q^j \in \mathbb{N}$. Man zeige: Es ist

$$|\mathcal{V}_c(\mathcal{P})| = \frac{\prod_{i=0}^{c}[d-i]_q}{\prod_{i=0}^{c}[i]_q},$$

insbesondere besitzt $\mathcal{P}$ genau $[d]_q$ Punkte und genau $\frac{[d]_q[d-1]_q}{q+1}$ Geraden; außerdem gehen durch jeden Punkt genau $[d-1]_q$ Geraden. Welche bekannten Aussagen erhält man für den Fall $d = 2$ bzw. für den Fall $\mathcal{P} = \mathbb{P}^d(\mathbb{F}_q)$?

(7.35) Aufgabe: Hyperebenen in allgemeiner Lage.
Es sei $\mathcal{P}$ ein endlicher projektiver Raum, der (P4) erfüllt, mit Punktmenge $\mathcal{V}$, Dimension $d := \dim(\mathcal{P}) \geq 0$ und Ordnung $q \geq 2$. Weiter seien $\mathcal{H}_0, \ldots, \mathcal{H}_d \leq \mathcal{V}$ Hyperebenen **in allgemeiner Lage**, das heißt, es gilt $\bigcap_{i=0}^{d} \mathcal{H}_i = \emptyset$. Man zeige: Es gilt $|\mathcal{V} \setminus (\bigcup_{i=0}^{d} \mathcal{H}_i)| = (q-1)^d$.

(7.36) Aufgabe: Automorphismen projektiver Räume.
a) Es seien $\mathbb{F}$ ein Körper und $d \geq 1$. Man zeige: Ist $A \in \mathrm{GL}_{d+1}(\mathbb{F})$, so induziert die natürliche Abbildung $\mathbb{F}^{(d+1)\times 1} \to \mathbb{F}^{(d+1)\times 1} \colon v \mapsto Av$ einen Automorphismus $\alpha_A \colon \mathbb{P}^d(\mathbb{F}) \to \mathbb{P}^d(\mathbb{F})$. Ist außerdem $B \in \mathrm{GL}_{d+1}(\mathbb{F})$, so gilt genau dann $\alpha_A = \alpha_B$, wenn $B = \lambda A$ für ein $0 \neq \lambda \in \mathbb{F}$ ist.
b) Daraus folgere man: Die Abbildung $\mathrm{GL}_{d+1}(\mathbb{F}) \to \mathrm{Aut}(\mathbb{P}^d(\mathbb{F})) \colon A \mapsto \alpha_A$ ist ein Gruppenhomomorphismus mit Kern $Z := \{\lambda E_{d+1}; 0 \neq \lambda \in \mathbb{F}\}$; die Gruppe $\mathrm{PGL}_{d+1}(\mathbb{F}) := \mathrm{GL}_{d+1}(\mathbb{F})/Z$ heißt **projektive volle lineare Gruppe**, sie bewirkt die **linearen** Automorphismen von $\mathbb{P}^d(\mathbb{F})$. Sind alle Automorphismen der projektiven Gerade $\mathbb{P}^1(\mathbb{F})$ linear?
c) Die Geraden der projektiven Ebene $\mathbb{P}^2(\mathbb{F})$ haben unter Verwendung des natürlichen Isomorphismus $\mathbb{P}^2(\mathbb{F}) \to \mathbb{P}^2(\mathbb{F})^*$ bekanntlich eine Parameterdarstellung. Mit welcher Matrix operiert α_A auf diesen Parameterdarstellungen?

(7.37) Aufgabe: Automorphismen von $\mathbb{P}^2(2)$.
Man zeige: Es ist $\mathrm{GL}_3(2) \cong \mathrm{PGL}_3(2)$, und alle Automorphismen von $\mathbb{P}^2(2)$ sind linear. Daraus folgere man: Es gilt $\mathrm{Aut}(\mathbb{P}^2(2)) \cong \mathrm{GL}_3(2)$.

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