On a mysterious partition identity

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1 Introduction

(1.1) Notation. Let $\mathcal{P}_n$ denote the set of all partitions of $n \in \mathbb{N}_0$. For $\lambda \in \mathcal{P}_n$ let $l(\lambda) \in \mathbb{N}_0$ be its length, i. e. the number of its non-zero parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{l(\lambda)} > 0$. Furthermore, let $s(\lambda) := n - l(\lambda) \in \mathbb{N}_0$ be its generalized sign, thus we have $\mathrm{sgn}(\lambda) = (-1)^{s(\lambda)}$. We also write $\lambda = [1^{a_1(\lambda)}, \ldots, n^{a_n(\lambda)}]$, where $a_i(\lambda) \in \mathbb{N}_0$.

Let $\mathcal{S}_n$ denote the symmetric group on $n \in \mathbb{N}_0$ letters. For $\lambda \in \mathcal{P}_n$ let $C_\lambda \subseteq \mathcal{S}_n$ denote the conjugacy class of elements of cycle type λ. For $n \in \mathbb{N}_0$ let $g_{[n]} := (1, \ldots, n) \in \mathcal{S}_n$, and for $\lambda \in \mathcal{P}_n$ let

$$N_\lambda := |\{g \in C_{[n]}; g \cdot g_{[n]} \in C_\lambda\}| \in \mathbb{N}_0.$$

Furthermore, for $s \in \mathbb{N}$ let

$$H(s) := \sum_{n \geq 1} \left(\sum_{\lambda \in \mathcal{P}_n, s(\lambda)=s, a_1(\lambda)=0} (-1)^{l(\lambda)} \cdot \frac{1}{n} \cdot \left(\prod_{i \geq 1} i^{a_i(\lambda)} \right) \cdot N_\lambda \right).$$

Note that for $\lambda \in \mathcal{P}_n$ such that $a_1(\lambda) = 0$ we have $n \leq 2 \cdot s(\lambda)$. Hence the outer sum occurring in the definition of $H(s)$ indeed is finite. If $s(\lambda)$ is odd, then we obviously have $N_\lambda = 0$, and hence for s odd we have $H(s) = 0$. The main result of this note is that this holds in general:

(1.2) Theorem. Let $s \in \mathbb{N}$. Then we have $H(s) = 0$.

2 Proof

The proof of Theorem (1.2), which will be given in Sections (2.4), (2.5) and (2.6), uses both character theoretic and combinatorial techniques. The necessary auxiliary results are stated in Propositions (2.2) and (2.3), respectively. As general references, the reader may consult [3] and [4], respectively. In particular, we use the usual combinatorial conventions for binomial coefficients, see [4, Ch.1.1]. We need some more

(2.1) Notation. For $\lambda \in \mathcal{P}_n$ and $\rho \in \mathcal{P}_m$ we write $\rho \trianglelefteq \lambda$ if $a_i(\rho) \leq a_i(\lambda)$ for $i \in \mathbb{N}$. In this case let $\lambda \setminus \rho := [1^{a_1(\lambda)-a_1(\rho)}, \ldots, n^{a_n(\lambda)-a_n(\rho)}] \in \mathcal{P}_{n-m}$. Thus we have $l(\lambda) = l(\rho) + l(\lambda \setminus \rho)$ and $s(\lambda) = s(\rho) + s(\lambda \setminus \rho)$.

For $g_\lambda \in C_\lambda$ let $C_{\mathcal{S}_n}(g_\lambda) \leq \mathcal{S}_n$ denote the centralizer of g_λ in $\mathcal{S}_n$ and let, see [3, La.1.2.15],

$$|C_{\mathcal{S}_n}(\lambda)| := |C_{\mathcal{S}_n}(g_\lambda)| = \left(\prod_{i\geq 1} i^{a_i(\lambda)}\right) \cdot \left(\prod_{i\geq 1} a_i(\lambda)!\right).$$

(2.2) Proposition. Let $\lambda \in \mathcal{P}_n$. Then we have

$$N_\lambda = \sum_{m=0}^{n} \left(\sum_{\rho \in \mathcal{P}_m, \rho \trianglelefteq \lambda} \frac{\operatorname{sgn}(\rho)}{n+1} \cdot \frac{m!}{|C_{\mathcal{S}_m}(\rho)|} \cdot \frac{(n-m)!}{|C_{\mathcal{S}_{n-m}}(\lambda \setminus \rho)|} \right).$$

Proof. As we have $C_\lambda^{-1} = C_\lambda$ and $C_{[n]}^{-1} = C_{[n]}$, by [2, Thm.2.4] we have $N_\lambda = c_{\lambda,[n],[n]}$, where $c_{\lambda,[n],[n]} \in \mathbb{N}_0$ denotes the central structure constant for the conjugacy class triple $(C_\lambda, C_{[n]}, C_{[n]})$ of $\mathcal{S}_n$.

For $\mu \in \mathcal{P}_n$ let χ_μ denote the corresponding irreducible ordinary character of $\mathcal{S}_n$, see [3, Thm.2.1.11]. For $\lambda \in \mathcal{P}_n$ let $\chi_\mu(\lambda) := \chi_\mu(g_\lambda)$, where $g_\lambda \in C_\lambda$. By [2, Exc.3.9], as χ_μ is real-valued we have

$$c_{\lambda,[n],[n]} = \frac{n!}{|C_{\mathcal{S}_n}(\lambda)| \cdot |C_{\mathcal{S}_n}([n])|} \cdot \sum_{\mu \in \mathcal{P}_n} \frac{\chi_\mu(\lambda) \cdot \chi_\mu([n])^2}{\chi_\mu(1)}.$$

By [3, 2.3.17] we have $\chi_\mu([n]) \neq 0$ if and only if $\mu = [n-m, 1^m] \in \mathcal{P}_n$, for $m \in \{0, \ldots, n-1\}$, is a hook partition. In this case we have $\chi_{[n-m,1^m]}([n]) = (-1)^m$. Furthermore, a straightforward calculation using [3, Thm.2.3.21] shows that $\chi_{[n-m,1^m]}(1) = \binom{n-1}{m}$. As $|C_{\mathcal{S}_n}([n])| = n$, this yields

$$c_{\lambda,[n],[n]} = \frac{(n-1)!}{|C_{\mathcal{S}_n}(\lambda)|} \cdot \sum_{m=0}^{n-1} \frac{1}{\binom{n-1}{m}} \cdot \chi_{[n-m,1^m]}(\lambda).$$

Using the determinantal form of the irreducible ordinary characters, see [3, Thm.2.3.15], for $m \geq 1$ we obtain

$$\chi_{[n-m,1^m]} = \left(\chi_{[n-m]} \otimes \chi_{[1^m]}\right)_{\mathcal{S}_{n-m} \times \mathcal{S}_m}^{\mathcal{S}_n} - \chi_{[n-(m-1),1^{m-1}]},$$

where $\otimes$ denotes the outer tensor product, and $(\cdot)_{\mathcal{S}_{n-m} \times \mathcal{S}_m}^{\mathcal{S}_n}$ denotes induction from the Young subgroup $\mathcal{S}_{n-m} \times \mathcal{S}_m \leq \mathcal{S}_n$ to $\mathcal{S}_n$. Note that $\chi_{[n-0,1^0]} = \chi_{[n]}$ is the trivial character and that $\chi_{[n-(n-1),1^{n-1}]} = \chi_{[1^n]}$ is the sign character of

$\mathcal{S}_n$. Furthermore, using the induction formula for class functions, see [2, p.64], for $\lambda \in \mathcal{P}_n$ we have

$$\left(\chi_{[n-m]} \otimes \chi_{[1^m]}\right)_{\mathcal{S}_{n-m}\times\mathcal{S}_m}^{\mathcal{S}_n}(\lambda) = \sum_{\rho\in\mathcal{P}_m,\rho\trianglelefteq\lambda} \frac{\mathrm{sgn}(\rho)\cdot|C_{\mathcal{S}_n}(\lambda)|}{|C_{\mathcal{S}_m}(\rho)|\cdot|C_{\mathcal{S}_{n-m}}(\lambda\setminus\rho)|}.$$

Using the identity $\frac{n+1}{n}\cdot\frac{1}{\binom{n-1}{m}} = \frac{1}{\binom{n}{m}} + \frac{1}{\binom{n}{m+1}}$, we finally obtain

$$\begin{aligned}
c_{\lambda,[n],[n]} &= \tfrac{n!}{(n+1)\cdot|C_{\mathcal{S}_n}(\lambda)|}\cdot\left(\sum_{m=0}^{n-1}\tfrac{\chi_{[n-m,1^m]}(\lambda)}{\binom{n}{m}} + \sum_{m=1}^{n}\tfrac{\chi_{[n-(m-1),1^{m-1}]}(\lambda)}{\binom{n}{m}}\right)\\
&= \tfrac{n!}{(n+1)\cdot|C_{\mathcal{S}_n}(\lambda)|}\cdot\sum_{m=0}^{n}\tfrac{1}{\binom{n}{m}}\cdot\left(\chi_{[n-m]}\otimes\chi_{[1^m]}\right)_{\mathcal{S}_{n-m}\times\mathcal{S}_m}^{\mathcal{S}_n}\\
&= \sum_{m=0}^{n}\sum_{\rho\in\mathcal{P}_m,\rho\trianglelefteq\lambda}\tfrac{\mathrm{sgn}(\rho)}{n+1}\cdot\tfrac{m!}{|C_{\mathcal{S}_m}(\rho)|}\cdot\tfrac{(n-m)!}{|C_{\mathcal{S}_m}(\lambda\setminus\rho)|}.
\end{aligned}$$

♯

(2.3) Proposition.
a) Vandermonde Convolution. Let $n, m \in \mathbb{Z}$ and $l \in \mathbb{N}_0$. Then we have

$$\binom{n+m}{l} = \sum_{k=0}^{l}\binom{n}{k}\cdot\binom{m}{l-k}.$$

b) Suranyi's Formula. Let $n, m \in \mathbb{Z}$ and $l \in \mathbb{N}_0$. Then we have

$$\sum_{k=0}^{l}(-1)^{l-k}\cdot\binom{n+m+1}{l-k}\cdot\binom{n+k}{k}\cdot\binom{m+k}{k} = \binom{n}{l}\cdot\binom{m}{l}.$$

c) Let $n, l \in \mathbb{N}$ such that $l \leq n$. Then we have

$$\sum_{\lambda\in\mathcal{P}_n,l(\lambda)=l}\frac{1}{\prod_{i\geq 1}a_i(\lambda)!} = \frac{1}{l!}\cdot\binom{n-1}{l-1}.$$

Proof. a) See [4, Ex.I.1.2.3].
b) By applying the Vandermonde Convolution, the assertion follows from the variant given in [4, Ch.4.4, p.144, l.5]; see also [1, 6.48].
c) See [4, Ch.5.5, p.183, l.5]. ♯

(2.4) Proof of Theorem (1.2). Let $s \in \mathbb{N}$ be fixed. For $\sigma = [\sigma_1, \ldots, \sigma_{l(\sigma)}] \in \mathcal{P}_s$ let $\sigma^+ := [\sigma_1 + 1, \ldots, \sigma_{l(\sigma)} + 1] \in \mathcal{P}_{s+l(\sigma)}$. Hence we have $l(\sigma^+) = l(\sigma)$ and thus $s(\sigma^+) = s(\sigma) = s$. Furthermore, we have $a_1(\sigma^+) = 0$ and $a_i(\sigma^+) = a_{i-1}(\sigma)$ for $i \geq 2$. Conversely, for $\lambda = [\lambda_1, \ldots, \lambda_{l(\lambda)}] \in \mathcal{P}_n$ such that $a_1(\lambda) = 0$ let $\lambda^- := [\lambda_1 - 1, \ldots, \lambda_{l(\lambda)} - 1] \in \mathcal{P}_{n-l(\lambda)}$. Hence we have $l(\lambda^-) = l(\lambda)$ and thus $s(\lambda^-) = s(\lambda)$. Furthermore, we have $a_i(\lambda^-) = a_{i+1}(\lambda)$ for $i \geq 1$. It is easily seen that these operations are inverse to each other. Hence the set of partitions to be summed over in the definition of $H(s)$ is given as $\{\sigma^+; \sigma \in \mathcal{P}_s\}$, thus we have

$$H(s) = \sum_{\sigma \in \mathcal{P}_s} \left(\frac{(-1)^{l(\sigma)}}{s + l(\sigma)} \cdot \left(\prod_{i \geq 1} i^{a_i(\sigma^+)} \right) \cdot N_{\sigma^+} \right).$$

Let $\sigma^+ \in \mathcal{P}_{s+l(\sigma)}$. If $\rho \in \mathcal{P}_m$ such that $\rho \trianglelefteq \sigma^+$, then we have $a_1(\rho) = 0$ and hence $\rho = \tau^+$ for some $\tau \in \mathcal{P}_{m-l(\rho)}$ such that $\tau \trianglelefteq \sigma$. Conversely, for each $\tau \in \mathcal{P}_t$ such that $\tau \trianglelefteq \sigma$ we obtain $\tau^+ \in \mathcal{P}_{t+l(\tau)}$ such that $\tau^+ \trianglelefteq \sigma^+$. Hence by Proposition (2.2) we obtain

$$H(s) = \sum_{\sigma \in \mathcal{P}_s} \left(\frac{(-1)^{l(\sigma)}}{(s + l(\sigma)) \cdot (s + l(\sigma) + 1)} \cdot \left(\prod_{i \geq 1} i^{a_i(\sigma^+)} \right) \right.$$

$$\left. \cdot \sum_{t=0}^{s} \sum_{\tau \in \mathcal{P}_t, \tau \trianglelefteq \sigma} \operatorname{sgn}(\tau^+) \cdot \frac{(t + l(\tau))!}{|C_{\mathcal{S}_{t+l(\tau)}}(\tau^+)|} \cdot \frac{(s + l(\sigma) - t - l(\tau))!}{|C_{\mathcal{S}_{s+l(\sigma)-t-l(\tau)}}(\sigma^+ \setminus \tau^+)|} \right).$$

Using the formula for centralizer orders given in Notation (2.1) and changing the order of summation yields

$$H(s) = \sum_{t=0}^{s} \sum_{\pi \in \mathcal{P}_t} \sum_{\pi' \in \mathcal{P}_{s-t}} \left(\frac{(-1)^{s-t} \cdot (t + l(\pi))! \cdot (s - t + l(\pi'))!}{(s + l(\pi) + l(\pi')) \cdot (s + l(\pi) + l(\pi') + 1)} \right.$$

$$\left. \cdot \frac{\operatorname{sgn}(\pi)}{\prod_{i \geq 1} a_i(\pi)!} \cdot \frac{\operatorname{sgn}(\pi')}{\prod_{i \geq 1} a_i(\pi')!} \right).$$

Keeping t fixed, we split the sum over all $\pi \in \mathcal{P}_t$ into sums over $\pi \in \mathcal{P}_t$ such that $l(\pi) = l$ is fixed, where in turn $l \in \{1, \ldots, t\}$. Note that for $\pi \in \mathcal{P}_t$ such that $l(\pi) = l$ we have $\operatorname{sgn}(\pi) = (-1)^{t-l}$. Analogously, the sum over $\pi' \in \mathcal{P}_{s-t}$ is split. Dealing with the cases $t = 0$ and $t = s$ separately, using Proposition (2.3) we obtain

$$H(s) = \sum_{t=1}^{s-1} \sum_{l=1}^{t} \sum_{l'=1}^{s-t} \frac{(-1)^{t+l+l'} \cdot (t + l)! \cdot (s - t + l')!}{(s + l + l') \cdot (s + l + l' + 1) \cdot l! \cdot (l')!} \cdot \binom{t-1}{l-1} \cdot \binom{s-t-1}{l'-1}$$

$$+ \sum_{l'=1}^{s} \frac{(-1)^{l'} \cdot (s + l')!}{(s + l') \cdot (s + l' + 1) \cdot (l')!} \cdot \binom{s-1}{l'-1}$$

$$+ \sum_{l=1}^{s} \frac{(-1)^{s+l} \cdot (s + l)!}{(s + l) \cdot (s + l + 1) \cdot l!} \cdot \binom{s-1}{l-1}$$

Using Propositions (2.5) and (2.6), which are proved below, and the notation introduced there, we finally obtain

$$\begin{aligned} H(s) &= H_0(s) + (-1)^s \cdot H_0(s) + \textstyle\sum_{t=1}^{s-1} (-1)^t \cdot H_t(s) \\ &= H_0(s) \cdot \left(\textstyle\sum_{t=0}^{s} (-1)^t \cdot \binom{s}{t} \right) \\ &= 0. \end{aligned}$$

♯

(2.5) Proposition. For $s \in \mathbb{N}$ let

$$H_0(s) := \sum_{l=0}^{s-1} \frac{(-1)^{l+1} \cdot (s+l)!}{(s+l+2) \cdot (l+1)!} \cdot \binom{s-1}{l}.$$

Then we have

$$H_0(s) = (-1)^s \cdot \frac{(s+1)! \cdot s! \cdot (s-1)!}{(2s+1)!}.$$

Proof. For $s \in \mathbb{N}$ and $k, m \in \mathbb{N}_0$ let

$$f_{k,m}(s) := \sum_{l=0}^{s-1} (-1)^{l+1} \cdot \frac{1}{s+l+m} \cdot \binom{s+l+k}{s-1} \cdot \binom{s-1}{l}.$$

Using both the identities $\frac{1}{s+l+m} = \frac{1}{s+m} \cdot (1 - \frac{l}{s+l+m})$ and $(-1)^{l+1} \cdot \binom{s+l+k}{s-1} = (-1)^k \cdot \binom{-s}{l+k+1}$, and applying the Vandermonde Convolution, see Proposition (2.3), yields

$$f_{k,m}(s) = \frac{1}{s+m} \cdot \left((-1)^s + \sum_{l=1}^{s-1} (-1)^l \cdot \frac{s+l+k}{s+l+m} \cdot \binom{s+l+k-1}{s-2} \cdot \binom{s-2}{l-1} \right).$$

Using the identity $\frac{s+l+k}{s+l+m} = 1 + \frac{k-m}{s+l+m}$, and again applying the Vandermonde Convolution, see Proposition (2.3), for $s \geq 2$ this yields

$$f_{k,m}(s) = \frac{k-m}{s+m} \cdot \sum_{l=0}^{s-2} \frac{(-1)^{l+1}}{s-1+l+m+2} \cdot \binom{s-1+l+k+1}{s-2} \cdot \binom{s-2}{l}.$$

Thus we have the recurrence $f_{k,m}(s) = \frac{k-m}{s+m} \cdot f_{k+1,m+2}(s-1)$, for $s \geq 2$. As we have $f_{k,m}(1) = \frac{-1}{m+1}$, iterating this recurrence shows

$$f_{k,m}(s) = f_{k+s-1,m+2s-2}(1) \cdot \prod_{i=0}^{s-2} \frac{k-m-i}{s+m+i} = \frac{-1}{2s+m-1} \cdot \frac{\binom{k-m}{s-1}}{\binom{2s+m-2}{s-1}}$$

Hence in particular we have $f_{0,2}(s) = \frac{-1}{2s+1} \cdot \frac{\binom{-2}{s-1}}{\binom{2s}{s-1}} = (-1)^s \cdot \frac{(s+1)! \cdot s!}{(2s+1)!}$. As we have $\frac{H_0(s)}{(s-1)!} = f_{0,2}(s)$, this proves the assertion. ♯

(2.6) Proposition. For $s, t \in \mathbb{N}$ such that $t < s$ let

$$H_t(s) := \sum_{l=1}^{t} \sum_{l'=1}^{s-t} \frac{(-1)^{l+l'} \cdot (t+l)! \cdot (s-t+l')!}{(s+l+l') \cdot (s+l+l'+1) \cdot l! \cdot (l')!} \cdot \binom{t-1}{l-1} \cdot \binom{s-t-1}{l'-1}.$$

Then we have

$$H_t(s) = \binom{s}{t} \cdot H_0(s).$$

Proof. For $s \in \mathbb{N}$ and $k, m \in \mathbb{N}_0$ let

$$g_{k,m}(s) := \sum_{l=0}^{s-1} (-1)^{l+1} \cdot \frac{1}{s+l+m} \cdot \binom{s+l+k}{s} \cdot \binom{s-1}{l}.$$

Using both the identities $\binom{s+l+k}{s} = \frac{s+l+k}{s} \cdot \binom{s+l+k-1}{s-1}$ and $\frac{s+l+k}{s+l+m} = 1 + \frac{k-m}{s+l+m}$, and applying the Vandermonde Convolution, see Proposition (2.3), we obtain $g_{k,m}(s) = \frac{(-1)^s}{s} + \frac{k-m}{s} \cdot f_{k-1,m}(s)$, where $f_{k,m}(s)$ is as in the proof of Proposition (2.5). A straightforward calculation shows

$$g_{k,m}(s) - g_{k,m+1}(s) = \frac{-(k+s)}{(s+m) \cdot (k-m)} \cdot \frac{\binom{k-m}{s}}{\binom{2s+m}{s}}.$$

We have

$$H_t(s) = t! \cdot (s-t)! \cdot \sum_{l=1}^{t} (-1)^l \cdot \binom{t+l}{l} \cdot \binom{t-1}{l-1}$$

$$\cdot \left(\sum_{l'=1}^{s-t} \cdot \frac{(-1)^{l'}}{(s+l+l') \cdot (s+l+l'+1)} \cdot \binom{s-t+l'}{l'} \cdot \binom{s-t-1}{l'-1} \right).$$

As $\frac{1}{(s+l+l') \cdot (s+l+l'+1)} = \frac{1}{(s+l+l')} - \frac{1}{(s+l+l'+1)}$, the inner sum is straightforwardly seen to be equal to $g_{1,t+l+1}(s-t) - g_{1,t+l+2}(s-t)$, hence we have

$$H_t(s) = t! \cdot (s-t-1)! \cdot \sum_{l=1}^{t} (-1)^{s+t+l} \cdot \frac{\binom{t+l}{l} \cdot \binom{t-1}{l-1} \cdot \binom{s+l-1}{s-t-1}}{\binom{2s-t+l+1}{s-t+1}}.$$

As $\binom{t+l}{l} \cdot \binom{s+l-1}{s-t-1} = \frac{(s+l-1) \cdot \dots \cdot (t+l+1)}{(s-t-1)!} \cdot \frac{(t+l) \cdot \dots \cdot (l+1)}{t!}$ this can be rewritten as

$$H_t(s) = (s-1)! \cdot \sum_{l=1}^{t} (-1)^{s+t+l} \cdot \frac{\binom{s+l-1}{s-1} \cdot \binom{t-1}{l-1}}{\binom{2s-t+l+1}{s-t+1}}.$$

Hence by Proposition (2.5) we have to show the following assertion

$$\sum_{l=1}^{t} (-1)^{t+l} \cdot \frac{\binom{s+l-1}{s-1} \cdot \binom{t-1}{l-1}}{\binom{2s-t+l+1}{s-t+1}} = \frac{\binom{s}{t}}{\binom{2s+1}{s+1}},$$

which is straightforwardly seen to be equivalent to the assertion

$$h_t(s) := \sum_{l=1}^{t} (-1)^{t+l} \cdot \frac{l}{t} \cdot \binom{2s+1}{t-l} \cdot \binom{s+l-1}{l} \cdot \binom{s+l}{l} = \binom{s}{t} \cdot \binom{s+1}{t}.$$

Using both the identities $\frac{l}{t} = 1 + \frac{l-t}{t}$ and $\binom{2s+1}{t-l} = \binom{2s}{t-l} + \binom{2s}{t-l-1}$, where as usual $\binom{2s}{-1} := 0$, we obtain

$$\begin{aligned} h_t(s) = {} & \sum_{l=1}^{t-1} (-1)^{t+l+1} \cdot \frac{2s+1}{t} \cdot \binom{2s}{t-1-l} \cdot \binom{s-1+l}{l} \cdot \binom{s+l}{l} \\ & + \sum_{l=1}^{t} (-1)^{t+l} \cdot \binom{2s}{t-l} \cdot \binom{s-1+l}{l} \cdot \binom{s+l}{l} \\ & + \sum_{l=1}^{t-1} (-1)^{t+l} \cdot \binom{2s}{t-1-l} \cdot \binom{s-1+l}{l} \cdot \binom{s+l}{l}. \end{aligned}$$

Finally, using Suranyi's Formula, see Proposition (2.3), by a straightforward calculation we obtain $h_t(s) = \binom{s}{t} \cdot \binom{s+1}{t}$. ♯

References

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