Green vertices and sources of simple modules of the symmetric group labelled by hook partitions

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Abstract

We determine the Green vertices and sources of many of the simple modular representations of the finite symmetric group being parametrized by hook partitions.
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1 Introduction and results

In recent years, one of the leading themes in the representation theory of finite groups has been the question how far the p-modular representation theory of a given group is determined by local data, i. e. by its p-subgroups and their normalisers. In this framework, it is immediate to ask for Green vertices and sources of simple modules. Astonishingly enough, only little seems to be known as far as particular examples are concerned. The present work is concerned with Green vertices and sources of simple modules of the symmetric group parametrised by hook partitions. In particular, we are able to deal with the natural module, which is the non-trivial constituent of the natural permutation representation. This work was begun in [13], and further related results can be found in [14].

The present paper is organised as follows: In Section 1 we fix some notation and state our results, where in (1.2) and (1.3) we deal with vertices and sources for general simple modules parametrised by hook partitions, while in (1.4) and (1.5) we consider the natural module in even characteristic. In Section 2 we give the proofs, subject to some propositions, which are subsequently proved in Sections 3 and 4. We assume the reader to be familiar with the representation theory of the symmetric group, see e. g. [8, 9], as well as with the theory of Green vertices and sources, see e. g. [4, Ch.III.4], [1, Ch.2.19].

(1.1) Let p be a rational prime and let F be a field of characteristic p. For $n \in \mathbb{N}_0$ let $n = a \cdot p + b \in \mathbb{N}$, where $a, b \in \mathbb{N}_0$ and $0 \leq b < p$. Let $\mathcal{P}_n$ be the set of partitions of n. A partition $\lambda = [n-r, 1^r] \in \mathcal{P}_n$, for $0 \leq r < n$, is called a hook partition. A partition $\lambda \in \mathcal{P}_n$ is called p-regular, if each of its parts occurs with multiplicity strictly less than p. A hook partition $\lambda = [n-r, 1^r] \in \mathcal{P}_n$ is p-regular if and only if either $r < p-1$ or $r = p-1$ and $n > p$.

For $n \in \mathbb{N}$ let $\mathcal{S}_n$ be the symmetric group on n letters. The field F is a splitting field of $F[\mathcal{S}_n]$, and the simple $F[\mathcal{S}_n]$-modules D^λ are parametrised by the p-regular partitions $\lambda \in \mathcal{P}_n$. In particular, $D^{[n]} \cong F_{\mathcal{S}_n}$ is the trivial $F[\mathcal{S}_n]$-module, and for $(n,p) \neq (2,2)$ the module $D := D^{[n-1,1]}$ is called the natural $F[\mathcal{S}_n]$-module. For any $\lambda \in \mathcal{P}_n$ let S^λ be the associated Specht $F[\mathcal{S}_n]$-module, and finally let $P_n \leq \mathcal{S}_n$ be a Sylow p-subgroup of $\mathcal{S}_n$. We are now prepared to state our results on simple $F[\mathcal{S}_n]$-modules being parametrised by hook partitions:

(1.2) Theorem. Let $\lambda = [n-r, 1^r] \in \mathcal{P}_n$ be a p-regular hook partition.
a) If $b = 0$ and $r < p-1$, then P_n is a vertex of D^λ.
b) If $r < b$, then P_n is a vertex of D^λ.
c) If $1 \leq b \leq r$, then $P_{(a-1)p}$ is a vertex of D^λ, and also a block defect group of the p-block of $F[\mathcal{S}_n]$ containing D^λ.

(1.3) Theorem. Let $\lambda = [n-r, 1^r] \in \mathcal{P}_n$ be a p-regular hook partition.
a) If $b \geq 1$, then D^λ is a trivial source module.
b) If p is odd, $b = 0$ and $r = 1$, then the restriction $\operatorname{res}_{P_n}(D)$ of the natural module D to $F[P_n]$ is a source of D.

A few comments are in order: An indecomposable module of a finite group is called a trivial source module, if it has the trivial module of one of its vertices as a source. For the cases in (1.2)a),b) by [4, Cor.III.6.8] the Sylow p-subgroups also are the block defect groups of the p-block of $F[\mathcal{S}_n]$ containing D^λ, hence in these cases the block defect groups coincide with the vertices of D^λ. If $b \geq 1$, then by [8, Thm.23.7] we have $D^\lambda \cong S^\lambda$, and the assertions in (1.2)b),c) and (1.3)a) also follow from [12]. Still, in (2.5) and (2.7) we include the proofs already given in [13], avoiding the more involved machinery used in [12].

Now we consider one of the cases left open in (1.2) and (1.3), namely the case $p = 2$, $b = 0$ and $r = p - 1 = 1$, i. e. the natural module $D = D^{[n-1,1]}$:

(1.4) Theorem. Let $p = 2$ and $n \in \mathbb{N}$ be even.
a) If $n \geq 6$, then P_n is a vertex of the natural module D.
b) If $n = 4$, then the normal subgroup $V_4 \trianglelefteq S_4$ is the vertex of $D = D^{[3,1]}$.

(1.5) Theorem. Let $p = 2$ and $n \in \mathbb{N}$ be even.
a) If $n \geq 6$, then $\operatorname{res}_{P_n}(D)$ is a source of the natural module D.
b) If $n = 4$, then $D = D^{[3,1]}$ is a trivial source module.

Again a few comments are in order: For $n \geq 6$ the Sylow 2-subgroups are the block defect groups of the 2-block containing D, hence again the block defect groups coincide with the vertices of D. But for the case $n = 4$ the vertex is a proper subgroup of the Sylow 2-subgroups, which still are the relevant block defect groups. Note that the assertions in (1.4)b) and (1.5)b) also follow from [12], or can be checked by inspection; we include a proof for completeness.

As far as p-regular hook partitions are concerned, it remains to determine the vertices of D^λ in the cases $b = 0$, where $n \neq p \neq 2$ and $r = p - 1$, as well as to determine the sources of D^λ in the cases $b = 0$, where $n \neq p \neq 2$ and $r > 1$. We have not been able to settle these cases, except for the special case in (2.6), but we conjecture the following:

(1.6) Conjecture. Let p be odd, let $b = 0$ and $n \neq p$.
a) Let $r = p - 1$. Then P_n is a vertex of D^λ.
b) Let $r > 1$. Then $\mathrm{res}_{P_n}(D^\lambda)$ is a source of D^λ.

Using the computer algebra system GAP [5] and applying methods from computational representation theory, this has been verified for $p = 3$ and $n \leq 18$ in [14]. Moreover, part a) is known to hold for the case $a \not\equiv 1 \pmod{p}$, a proof will appear in [2].

2 Proofs

(2.1) Let $\mathcal{S}_n$ act naturally on the set $\Omega := \{1, \ldots, n\}$. For a partition $\lambda = [\lambda_1, \ldots, \lambda_l] \in \mathcal{P}_n$ let $\Omega = \coprod_{j=1}^{l} \Omega_j$, where $\Omega_j := \{(\sum_{k=1}^{j-1} \lambda_k)+1, \ldots, (\sum_{i=k}^{j} \lambda_k)\}$, for $1 \leq j \leq l$. Let $\mathcal{S}_\lambda := \prod_{j=1}^{l} \mathcal{S}_{\Omega_j} \leq \mathcal{S}_n$ be the corresponding Young subgroup, and let M^λ denote the associated permutation $F[\mathcal{S}_n]$-module.

(2.2) We proceed to state and prove (2.3), aiming to relate $D^{[n-r,1^r]}$ to D, where $[n-r, 1^r] \in \mathcal{P}_n$ is p-regular. Actually, (2.3)a) is well-known by folklore, but since there is no suitable general reference known to us we include a proof; note that in [6] only the case $n = p$ for odd p is treated. Moreover, (2.3)b) is implicit in [11], but we include an explicit proof for convenience:

Given any partition $\lambda \in \mathcal{P}_n$, the Specht $F[\mathcal{S}_n]$-module $S^\lambda \leq M^\lambda$ is described as follows: Let T be a λ-tableau, and let $\mathcal{R}_T \leq \mathcal{S}_n$ and $\mathcal{C}_T \leq \mathcal{S}_n$ be the corresponding row and column stabilisers. The orbit $\overline{T}$ of T under the action of $\mathcal{R}_T$ is called the associated λ-tabloid. The F-vector space generated by the λ-tabloids is as an $F[\mathcal{S}_n]$-module isomorphic to M^λ. For a λ-tableau T let $e_T := \sum_{g \in \mathcal{C}_T} \mathrm{sgn}(g) \cdot \overline{T^g} \in M^\lambda$ denote the associated λ-polytabloid, where for $g \in \mathcal{S}_n$ we have $e_T \cdot g = e_{T^g}$. A λ-tableau being called standard if its entries are strictly increasing both along rows and columns, the set $\{e_T; T \text{ standard } \lambda\text{-tableau}\} \subseteq M^\lambda$ of standard λ-polytabloids is F-linearly independent, its F-span is $F[\mathcal{S}_n]$-invariant and called the Specht module S^λ.

For a hook partition $\lambda = [n-r, 1^r] \in \mathcal{P}_n$, where $0 \leq r < n$, a standard λ-tableau T is described by its first column entries $1 < t_1 < \ldots < t_r$, allowing to abbreviate T by $\langle 1, t_1, \ldots, t_r \rangle$. Thus $\{e_{\langle 1,t_1,\ldots,t_r\rangle}; 1 < t_1 < \ldots < t_r \leq n\}$ is an F-basis of $S^{[n-r,1^r]}$. In particular, $\{e_{\langle 1,t\rangle}; 1 < t \leq n\}$ is an F-basis of $S^{[n-1,1]}$, implying that $\{e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle}; 1 < t_1 < \ldots < t_r \leq n\}$ is an F-basis of the r-fold exterior power $\bigwedge^r S^{[n-1,1]}$, for $0 < r < n$.

(2.3) Proposition. Let $1 \leq r < n$ and $\lambda = [n-r, 1^r] \in \mathcal{P}_n$ be a hook partition.
a) The F-linear map $\bigwedge^r S^{[n-1,1]} \to S^\lambda \colon e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle} \mapsto e_{\langle 1,t_1,\ldots,t_r\rangle}$, for all $1 < t_1 < \ldots < t_r \leq n$, is an isomorphism of $F[\mathcal{S}_n]$-modules.
b) If $\lambda = [n-r, 1^r] \in \mathcal{P}_n$ is p-regular, then for the natural module $D = D^{[n-1,1]}$ we have $\bigwedge^r D \cong D^\lambda$ as $F[\mathcal{S}_n]$-modules.

Proof. a) As the elements $s_i := (i, i+1) \in \mathcal{S}_n$, where $1 \leq i < n$, generate $\mathcal{S}_n$, it suffices to compare $(e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle}) \cdot s_i \in \bigwedge^r S^{[n-1,1]}$ and $e_{\langle 1,t_1,\ldots,t_r\rangle} \cdot s_i \in S^\lambda$:

Let $i > 1$. Then we have $e_{\langle 1,i\rangle} \cdot s_i = e_{\langle 1,i+1\rangle}$ and $e_{\langle 1,i+1\rangle} \cdot s_i = e_{\langle 1,i\rangle}$, as well as $e_{\langle 1,t\rangle} \cdot s_i = e_{\langle 1,t\rangle}$ for $1 < t \leq n$ and $t \notin \{i, i+1\}$. If both $i, i+1 \notin \{t_1, \ldots, t_r\}$, then $(e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle}) \cdot s_i = e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle}$ and $e_{\langle 1,t_1,\ldots,t_r\rangle} \cdot s_i = e_{\langle 1,t_1,\ldots,t_r\rangle}$. If $i \in \{t_1, \ldots, t_r\}$ but $i+1 \notin \{t_1, \ldots, t_r\}$, then assuming $i = t_r$ for convenience we have $(e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_{r-1}\rangle} \wedge e_{\langle 1,i\rangle}) \cdot s_i = e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_{r-1}\rangle} \wedge e_{\langle 1,i+1\rangle}$ and $e_{\langle 1,t_1,\ldots,t_{r-1},i\rangle} \cdot s_i = e_{\langle 1,t_1,\ldots,t_{r-1},i+1\rangle}$. The case $i+1 \in \{t_1, \ldots, t_r\}$ but $i \notin \{t_1, \ldots, t_r\}$ is dealt with analogously. If both $i, i+1 \in \{t_1, \ldots, t_r\}$, then we have $(e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle}) \cdot s_i = -(e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle})$, and since in this case $s_i \in \mathcal{C}_{\langle 1,t_1,\ldots,t_r\rangle}$ we have $e_{\langle 1,t_1,\ldots,t_r\rangle} \cdot s_i = \mathrm{sgn}(s_i) \cdot e_{\langle 1,t_1,\ldots,t_r\rangle} = -e_{\langle 1,t_1,\ldots,t_r\rangle}$.

Let $i = 1$. Then we have $e_{\langle 1,2\rangle} \cdot s_1 = e_{\langle 2,1\rangle} = -e_{\langle 1,2\rangle}$, while for $2 < t \leq n$ using the Garnir relation for the set $\{2, t\} \,\dot\cup\, \{1\}$, see [8, Ch.7], we obtain $e_{\langle 1,t\rangle} \cdot s_1 = e_{\langle 2,t\rangle} = e_{\langle 1,t\rangle} - e_{\langle 1,2\rangle}$. If $t_1 > 2$ then $(e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle}) \cdot s_1 = (e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle}) - \sum_{k=1}^r (e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_{k-1}\rangle} \wedge e_{\langle 1,2\rangle} \wedge e_{\langle 1,t_{k+1}\rangle} \ldots \wedge e_{\langle 1,t_r\rangle})$, where the summands are $e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_{k-1}\rangle} \wedge e_{\langle 1,2\rangle} \wedge e_{\langle 1,t_{k+1}\rangle} \ldots \wedge e_{\langle 1,t_r\rangle} = (-1)^{k+1} \cdot (e_{\langle 1,2\rangle} \wedge e_{\langle 1,t_1\rangle} \wedge \ldots \wedge e_{\langle 1,t_{k-1}\rangle} \wedge e_{\langle 1,t_{k+1}\rangle} \ldots \wedge e_{\langle 1,t_r\rangle})$. Moreover, using the Garnir relation for the set $\{2, t_1, \ldots, t_r\} \,\dot\cup\, \{1\}$ we obtain $e_{\langle 1,t_1,\ldots,t_r\rangle} \cdot s_1 = e_{\langle 2,t_1,\ldots,t_r\rangle} = e_{\langle 1,t_1,\ldots,t_r\rangle} - \sum_{k=1}^r e_{\langle 1,t_1,\ldots,t_{k-1},2,t_{k+1},\ldots,t_r\rangle}$, where using the fact that $(2, t_1, \ldots, t_{k-1}) \in \mathcal{C}_{\langle 1,t_1,\ldots,t_{k-1},2,t_{k+1},\ldots,t_r\rangle}$, the summands can be rewritten as $e_{\langle 1,t_1,\ldots,t_{k-1},2,t_{k+1},\ldots,t_r\rangle} = (-1)^{k+1} \cdot e_{\langle 1,2,t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_r\rangle}$.

If finally $t_1 = 2$, then we analogously obtain $(e_{\langle 1,2\rangle} \wedge e_{\langle 1,t_2\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle}) \cdot s_1 = -(e_{\langle 1,2\rangle} \wedge e_{\langle 1,t_2\rangle} \wedge \ldots \wedge e_{\langle 1,t_r\rangle})$ and $e_{\langle 1,2,t_2,\ldots,t_r\rangle} \cdot s_1 = e_{\langle 2,1,t_2,\ldots,t_r\rangle} = -e_{\langle 1,2,t_2,\ldots,t_r\rangle}$.

b) If $b \geq 1$, then by [8, Thm.23.7] we have $D^\lambda = S^\lambda$ anyway. Thus we may assume $b = 0$. By [11], see also [8, Ch.11, Thm.24.1], and noting that $p = 2$ implies $r = 1$ anyway, S^λ is a uniserial $F[\mathcal{S}_n]$-module of composition length 2, having socle $\mathrm{soc}(S^\lambda) \cong D^{[n-r+1,1^{r-1}]}$ and head $S^\lambda/\mathrm{rad}(S^\lambda) = S^\lambda/\mathrm{soc}(S^\lambda) \cong D^\lambda$. In particular, we have $\mathrm{soc}(S^{[n-1,1]}) \cong F_{\mathcal{S}_n}$ and $S^{[n-1,1]}/\mathrm{rad}(S^{[n-1,1]}) \cong D$, facilitating an $F[\mathcal{S}_n]$-epimorphism $S^\lambda \cong \bigwedge^r S^{[n-1,1]} \to \bigwedge^r D$. Since $0 < \dim_F(\bigwedge^r D) = \binom{n-2}{r} < \binom{n-1}{r} = \dim_F(S^\lambda)$, by the structure of S^λ we conclude that the latter induces an isomorphism $D^\lambda \cong \bigwedge^r D$ of $F[\mathcal{S}_n]$-modules. ♯

(2.4) Remark. Let $M := M^{[n-1,1]}$ denote the natural permutation $F[\mathcal{S}_n]$-module, and let $\{\gamma_1, \ldots, \gamma_n\}$ denote its permutation F-basis. If $n > 1$ and $b \neq 0$, then we have $M \cong F_{\mathcal{S}_n} \oplus D$ as $F[\mathcal{S}_n]$-modules.

If $b = 0$, then M is uniserial. More precisely, the only proper submodules of M are $M' := \mathrm{rad}(M) = \{\sum_{i=1}^n a_i\gamma_i \in M; \sum_{i=1}^n a_i = 0\} \leq M$ and $M'' := \mathrm{soc}(M) = \langle \gamma^+ \rangle_F \leq M$, where $\gamma^+ := \sum_{i=1}^n \gamma_i \in M$, and we have $M'' \cong F_{\mathcal{S}_n} \cong M/M'$. Let $\bar{\ }\colon M \to M/M''$ be the natural epimorphism. We have $M'' \leq M'$, where equality holds if and only if $p = n = 2$, and for $n > 2$ we have $D \cong M'/M''$ as $F[\mathcal{S}_n]$-modules.

(2.5) Proof of (1.2).

a), b) By (2.3) and (2.4), for $b = 0$ and $r < p-1$ we have $\dim_F(D^\lambda) = \binom{n-2}{r} = \frac{\prod_{i=0}^{r-1}(n-2-i)}{r!}$, while for $r < b$, which implies $b \geq 1$, we have $\dim_F(D^\lambda) = \binom{n-1}{r} = \frac{\prod_{i=0}^{r-1}(n-1-i)}{r!}$. In these cases p does not divide the numerator of the above fractions, and thus p does not divide $\dim_F(D^\lambda)$ either. By [1, Thm.2.19C.26] we conclude that P_n is a vertex of D^λ.

c) Let $1 \leq b \leq r$. Again we have $\dim_F(D^\lambda) = \binom{n-1}{r} = \frac{\prod_{i=0}^{r-1}(n-1-i)}{r!}$. As $r < p$ the denominator of this fraction is not divisible by p, and $n-1-(b-1)$ is the only factor of the numerator being divisible by p, hence for the p-part of $\dim_F(D^\lambda)$ we have $(\dim_F(D^\lambda))_p = (n-b)_p$.

Let B be the p-block of $F[\mathcal{S}_n]$ containing D^λ, whose basic invariant by [9, Thm.6.1.21] is the p-core $\overline{\lambda}$ of the partition $\lambda = [n-r, 1^r] \in \mathcal{P}_n$. Since $b \neq 0$ and $r < p$, we have $\overline{\lambda} = [p+b-r, 1^r] \in \mathcal{P}_{p+b}$, and hence the p-weight of B equals $a-1$. By [9, Thm.6.2.39] a Sylow p-subgroup $P_{(a-1)p}$ of $\mathcal{S}_{(a-1)p} \leq \mathcal{S}_n$ is a defect group of B, where we have $|P_{(a-1)p}| = (\frac{n!}{ap})_p$. By [4, Cor.III.6.8] there is a subgroup $Q \leq P_{(a-1)p}$ being a vertex of D^λ. By [1, Thm.2.19C.26] the index $[P_n \colon Q]$ divides $(n-b)_p = (ap)_p$, and thus $(\frac{n!}{ap})_p$ divides $|Q|$. $\sharp$

(2.6) Remark. In the case $b = 0$, where $n \neq p \neq 2$, and $r = p-1$ we analogously observe that $(\frac{n!}{(a-1)p})_p$ divides the order $|Q|$ of a vertex Q of $D^{[n-p+1,1^{p-1}]}$. But since $\overline{\lambda} = [] \in \mathcal{P}_0$, the p-weight of the p-block B containing $D^{[n-p+1,1^{p-1}]}$ equals a, implying that the Sylow p-subgroups $P_{ap} = P_n \leq \mathcal{S}_n$ are the block defect groups of B. Hence applying the technique of (2.5) is in general not sufficient to determine the vertices of $D^{[n-p+1,1^{p-1}]}$.

Still, for the case $a = 2$, i. e. $n = 2p \neq 4$, and $r = p-1$ we may argue as follows: Assume that $Q < P_{2p}$, thus $|Q| = p$ and Q is cyclic. By [3] Q is a block defect group of B, a contradiction. Thus P_{2p} is a vertex of $D^{[p+1,1^{p-1}]}$. $\sharp$

We proceed to prove (1.3), (1.4) and (1.5). The proofs are given in (2.7) and (2.8), and depend on (3.7), (4.2) and (4.5), which are proved subsequently.

(2.7) Proof of (1.3).
a) Let $b \geq 1$, and we may assume $r \geq 1$. By (2.4) we have $M \cong F_{\mathcal{S}_n} \oplus D$ as $F[\mathcal{S}_n]$-modules. Now D being a direct summand of a permutation module, by [10, La.II.12.5] is a trivial source module. For $r \geq 2$ by (2.3) we have $\bigwedge^r M \cong \bigwedge^{r-1} D \oplus \bigwedge^r D \cong D^{[n-r+1,1^{r-1}]} \oplus D^\lambda$. The symmetric group $\mathcal{S}_r$ acts on the r-fold tensor product $M^{\otimes r}$ by permuting the tensor factors, while $\mathcal{S}_n$ acts diagonally, hence $M^{\otimes r}$ becomes an $F[\mathcal{S}_n \times \mathcal{S}_r]$-module. As $r < p$, applying the idempotent $\frac{1}{r!} \cdot \sum_{g \in \mathcal{S}_r} \mathrm{sgn}(g) \cdot g \in F[\mathcal{S}_r]$, which by [9, 5.2.18] projects $M^{\otimes r}$ onto $\bigwedge^r M$, shows that $\bigwedge^r M$ is an $F[\mathcal{S}_n]$-direct summand of the permutation module $M^{\otimes r}$. This shows that D^λ is a trivial source module.

b) Let p be odd, $b = 0$ and $r = 1$. Then by (1.2) P_n is a vertex of D. As by (3.7) $\mathrm{res}_{P_n}(D)$ is indecomposable, the assertion follows from [4, La.III.4.6]. $\sharp$

(2.8) Proof of (1.4) and (1.5).
a) Let $n \geq 6$. By [4, La.III.4.4] D is relatively P_n-projective. By (3.7) $\mathrm{res}_{P_n}(D)$ is indecomposable, and by [4, La.III.4.6] each vertex of $\mathrm{res}_{P_n}(D)$ is a vertex of D. Assume $\mathrm{res}_{P_n}(D)$ is relatively Q-projective for a maximal subgroup $Q < P_n$. Then by [4, La.III.4.2] there is an indecomposable direct summand V of $\mathrm{res}_Q(D)$ such that $\mathrm{res}_{P_n}(D)$ is a direct summand of the induced module $\mathrm{ind}_{P_n}(V)$. We may assume that the $F[Q]$-module V is absolutely indecomposable, otherwise by [4, La.I.18.7] we replace the field F by a suitable finite field extension. By Green's indecomposability theorem, see [4, Thm.III.3.8], we conclude $\mathrm{res}_{P_n}(D) \cong \mathrm{ind}_{P_n}(V)$ as $F[P_n]$-modules, implying $\dim_F(V) = \frac{\dim_F(D)}{2}$. But by (4.2) and (4.5) $\mathrm{res}_Q(D)$ does not have such an indecomposable direct summand, a contradiction, implying that P_n is a vertex of D. Finally, by [4, La.III.4.2] the sources of D are direct summands of $\mathrm{res}_{P_n}(D)$, and since the latter is indecomposable it is the source of D.

b) Let $n = 4$. The restriction $\mathrm{res}_{V_4}(D)$ of D to the normal subgroup $V_4 \trianglelefteq \mathcal{S}_4$ being semisimple, we have $\mathrm{res}_{V_4}(D) \cong F_{V_4} \oplus F_{V_4}$ as $F[V_4]$-modules. As $\mathcal{S}_4/V_4 \cong \mathcal{S}_3$ and $\dim_F(D) = 2$, the $F[\mathcal{S}_4]$-module D can be considered as a projective simple $F[\mathcal{S}_3]$-module, as such having the trivial subgroup of $\mathcal{S}_3$ as its vertex. By [4, Cor.III.4.13] we conclude that $V_4 \trianglelefteq \mathcal{S}_4$ is a vertex of D. Finally, by [4, La.III.4.2] again we see that F_{V_4} is the source of D. $\sharp$

3 Restriction to Sylow subgroups

(3.1) For a rational prime $p \in \mathbb{N}$ let $n = \sum_{i \in \mathbb{N}_0} b_i p^i \in \mathbb{N}$, be the p-adic expansion of $n \in \mathbb{N}$, where $0 \leq b_i < p$. Compared to the notation in (1.1) we have $b := b_0$ and $a := \frac{n-b_0}{p}$. Let $l := \sum_{i \in \mathbb{N}_0} b_i \in \mathbb{N}$ be the number of parts, and $n_1 \geq n_2 \geq \ldots \geq n_l$ be the sequence of p-powers occurring in the above p-adic expansion, i. e. the p-power p^i occurs b_i times in the sequence $n_1, n_2, \ldots, n_l$, for $i \in \mathbb{N}_0$. Note that the maximum p-power dividing the group order $|\mathcal{S}_n|$ is given as $\log_p((n!)_p) = \sum_{i \in \mathbb{N}_0} b_i \cdot \frac{p^i - 1}{p-1}$.

By [9, 4.1.24] a Sylow p-subgroup $P_n \leq \mathcal{S}_n$ is given as follows: Still letting $\Omega := \{1, \ldots, n\}$, we have $\Omega = \coprod_{j=1}^{l} \Omega_j$, where $\Omega_j := \{(\sum_{k=1}^{j-1} n_k)+1, \ldots, \sum_{k=1}^{j} n_k\} \subseteq \Omega$. Let P_{n_j} be a Sylow p-subgroup of $\mathcal{S}_{\Omega_j}$, for $1 \leq j \leq l$, where we make a specific choice for P_{n_j} in (3.2). Note that for p odd we might have $j \neq j'$ such that $n_j = n_{j'}$, but still we distinguish P_{n_j} and $P_{n_{j'}}$, which act non-trivially on different subsets of Ω, to avoid notational overload. Now we have $P_n \cong \prod_{j=1}^{l} P_{n_j}$, and we let $p_j \colon P_n \to P_{n_j}$ be the corresponding projections. Moreover, we have $P_{n_j} \cong (\cdots (C_p \wr C_p) \wr C_p) \cdots \wr C_p)$, where the right hand side is an $(j-1)$-fold iterated regular wreath product, C_p denoting the cyclic group of order p, and the 0-fold and (-1)-fold wreath products are defined as C_p and $\{1\}$, respectively.

(3.2) Let $n = p^m$, where $m \in \mathbb{N}_0$. Let $w_1 := 1 \in \mathcal{S}_1$, and for $m \geq 1$ let

$$w_n := \prod_{i=1}^{\frac{n}{p}} (i, i + \frac{n}{p}, i + \frac{2n}{p}, \ldots, i + \frac{(p-1)n}{p}) \in \mathcal{S}_n,$$

in particular $w_n \in \mathcal{S}_n$ has cycle type $[p^{\frac{n}{p}}]$. By (3.1) the set $\{w_{p^j}; 1 \leq j \leq m\}$ generates a Sylow p-subgroup of $P_n \leq \mathcal{S}_n$, which specifies P_n in the p-power case $n = p^m$. Moreover, let $y_1 := 1$, and for $m \geq 1$ let

$$y_n := w_n \cdot y_{\frac{n}{p}} = w_{p^m} \cdot w_{p^{m-1}} \cdot \cdots \cdot w_p \in P_n.$$

(3.3) Lemma. Let $m \in \mathbb{N}_0$ and $n = p^m$. Then $y_n \in \mathcal{S}_n$ has cycle type $[n]$.

Proof. We may assume that $m \geq 1$. We first order $\Omega = \{1, \ldots, n\}$ as follows: For $c \in \Omega$ let $c - 1 = \sum_{i=0}^{m-1} c_i p^i$ be the p-adic expansion of $c - 1$, where $0 \leq c_i < p$. This associates to any $c \in \Omega$ the coefficient vector $[c_0, \ldots, c_{m-1}]$ of length m. We order these vectors lexicographically, i. e. we get the sequence

$$[0, \ldots, 0, 0], [0, \ldots, 0, 1], \ldots, [0, \ldots, 0, p-1], [0, \ldots, 1, 0], \ldots, [p-1, \ldots, p-1, p-1],$$

which translates to order Ω as follows: $1, 1 + \frac{n}{p}, \ldots, 1 + \frac{(p-1)n}{p}, 1 + \frac{n}{p^2}, \ldots, n$, where for the sake of completeness we let 1 be the successor of n. We now show that $y_n = w_n \cdot y_{\frac{n}{p}} \in \mathcal{S}_n$ is an n-cycle respecting this ordering:

By induction we may assume that the assertion holds for $y_{\frac{n}{p}} \in \mathcal{S}_{\frac{n}{p}}$, with respect to the analogous ordering on $\{1, \ldots, \frac{n}{p}\}$ afforded by coefficient vectors of length $m - 1$. Since $y_{\frac{n}{p}} \in \mathcal{S}_n$ fixes all the elements of $\Omega \setminus \{1, \ldots, \frac{n}{p}\}$, i. e. those $c \in \Omega$ such that $c_{m-1} > 0$, we distinguish two cases: If $c_{m-1} < p - 1$, then we have $c \cdot y_n = (c \cdot w_n) \cdot y_{\frac{n}{p}} = (c + \frac{n}{p}) \cdot y_{\frac{n}{p}} = c + \frac{n}{p} \in \Omega$, which is the successor of $c \in \Omega$ in the above ordering. If $c_{m-1} = p - 1$, then $c \cdot y_n = (c \cdot w_n) \cdot y_{\frac{n}{p}} = (c - \frac{(p-1)n}{p}) \cdot y_{\frac{n}{p}} \in \Omega$, which by induction again is the successor of $c \in \Omega$ in the above ordering. ♯

(3.4) Let $Y_n := \langle y_n \rangle \cong C_n$. The trivial module is the only simple $F[Y_n]$-module up to isomorphism, and there are precisely n isomorphism types of indecomposable $F[Y_n]$-modules U_i, for $1 \leq i \leq n$, all of which are uniserial, see [1, Prop.2.20B.11]. They can be distinguished by their F-dimension, and we assume notation to be chosen such that $\dim_F(U_i) = i$. In particular, the regular module $U_n \cong F[Y_n]$ is the only projective indecomposable $F[Y_n]$-module. For the action of Y_n on U_i we have $\dim_F \ker_{U_i}(y_n - 1)^j = j$, for $1 \leq j \leq i$, thus the minimum polynomial of the action of y_n on U_i is $(T - 1)^i \in F[T]$.

(3.5) Lemma. Let $n = p^m$, where $m \in \mathbb{N}$. Then $\mathrm{res}_{Y_n}(D)$ is uniserial.

Proof. By (3.3) the element $y_n \in P_n$ has cycle type $[n]$. The minimum polynomial of the action of y_n on the natural permutation module $M := M^{[n-1,1]}$ is $T^n - 1 = (T - 1)^n \in F[T]$, and thus $\mathrm{res}_{Y_n}(M) \cong U_n$ is a uniserial $F[Y_n]$-module. As D is an $F[\mathcal{S}_n]$-constituent of M, the assertion follows. ♯

(3.6) Let again $n \in \mathbb{N}$ be arbitrary. Keeping the notation of (3.1), for the permutation module $M := M^{[n-1,1]}$ we have $\mathrm{res}_{P_n}(M) \cong \bigoplus_{j=1}^{l} \mathrm{res}_{P_{n_j}}(M^{[n_j-1,1]})$

as $F[P_n]$-modules, where the action on the j-th summand is given by the projection $p_j\colon P_n \to P_{n_j}$, for $1 \leq j \leq l$.

As in (3.2) let $y_{n_j} \in P_{n_j} \leq \mathcal{S}_{n_j}$ and $Y_{n_j} := \langle y_{n_j} \rangle \leq P_{n_j}$, for $1 \leq j \leq l$, where again for p odd we might have $j \neq j'$ such that $n_j = n_{j'}$, but again we distinguish Y_{n_j} and $Y_{n_{j'}}$. Letting $Y_n := \prod_{j=1}^{l} Y_{n_j} \leq P_n \leq \mathcal{S}_n$, from the proof of (3.5) we get $\mathrm{res}_{Y_n}(M) \cong \bigoplus_{j=1}^{l} \mathrm{res}_{Y_{n_j}}(M^{[n_j-1,1]}) \cong \bigoplus_{j=1}^{l} F[Y_{n_j}]$, where again the action on the j-th summand is given by the projection $p_j|_{Y_n}\colon Y_n \to Y_{n_j}$.

(3.7) Proposition. Let $b = 0$. Then $\mathrm{res}_{P_n}(D)$ is indecomposable.

Proof. By (3.5) we may assume that $n \in \mathbb{N}$ is not a p-power, i. e. the p-adic expansion of n has $l \geq 2$ parts. Let $v \in M$ such that $0 \neq \overline{v} \in \mathrm{soc}(\mathrm{res}_{Y_n}(M/M''))$, where $\overline{}$ is as in (2.4). For $1 \leq j \leq l$ we have $vy_{n_j} = v + a \cdot \gamma^+$, for some $a \in F$. As Y_{n_j} fixes $\Omega \setminus \Omega_j \neq \emptyset$ elementwise, it follows that $a = 0$ and $vy_{n_j} = v$. Letting $\gamma_j^+ := \sum_{i \in \Omega_j} \gamma_i \in M'$, for $1 \leq j \leq l$, we have $\gamma^+ = \sum_{j=1}^{l} \gamma_j^+$, and $\{\overline{\gamma_j^+} \in D; 1 \leq j \leq l-1\}$ is an F-basis of $\mathrm{soc}(\mathrm{res}_{Y_n}(D))$.

For $1 \leq j \leq l-1$ let $v_j := (\gamma_{n_1+n_2+\ldots+n_j}) - (\gamma_{n_1+n_2+\ldots+n_j+1}) \in M'$, and let $v_l := \gamma_n - \gamma_1 \in M'$. Letting $y_{n_j}^+ := (y_{n_j} - 1)^{n_j-1} = \sum_{i=0}^{n_j-1} y_{n_j}^i \in F[Y_{n_j}] \subseteq F[Y_n]$, for $1 \leq j \leq l$, we have $y_{n_j}^+ \cdot (y_{n_j} - 1) = 0 \in F[Y_n]$ and $v_j y_{n_j}^+ = \gamma_j^+ \notin M''$, hence $\overline{v_j} y_{n_j}^+ = \overline{\gamma_j^+} \neq 0 \in D$. As for the action of Y_{n_j} on M we have $\dim_F \ker_M(y_{n_j}^+) = n-1$, we conclude that $\dim_F \ker_{M'}(y_{n_j}^+) \geq \dim_F(M')-1$. For the action of Y_{n_j} on $D \cong M'/M''$ we conclude $\dim_F \ker_D(y_{n_j}^+) \geq \dim_F(D) - 1$, and $\dim_F \ker_D(y_{n_j}^+) = \dim_F(D) - 1$, for $1 \leq j \leq l$.

We fix a direct sum decomposition of $\mathrm{res}_{Y_n}(D)$. For $1 \leq j \leq l$ let $V_j \leq \mathrm{res}_{Y_n}(D)$ be the uniquely determined summand on which $y_{n_j}^+$ does not act by the zero map, where the V_j need not be pairwise distinct. Indeed, we have $\overline{\gamma_j^+} \in V_j$, and from $\sum_{j=1}^{l} \overline{\gamma_j^+} = \overline{\gamma^+} = 0 \in D$ we conclude that $V_1 = V_2 = \ldots = V_l$ holds. Thus $\mathrm{soc}(\mathrm{res}_{Y_n}(D)) \leq V_1 = V_2 = \ldots = V_l$, and even $\mathrm{res}_{Y_n}(D)$ is indecomposable. $\sharp$

4 The case $p = 2$

(4.1) Let $p = 2$ and let $n = 2^m \in \mathbb{N}$ for some $m \in \mathbb{N}_0$. We keep the notation of (3.2), and for $m \geq 1$ let

$$x_n := y_n^2 = (w_n^{-1} \cdot y_{\frac{n}{2}} \cdot w_n) \cdot y_{\frac{n}{2}} \in P_n.$$

As $w_n \in \mathcal{S}_n$ interchanges $\{1, \ldots, \frac{n}{2}\}$ and $\{\frac{n}{2}+1, \ldots, n\}$, we by (3.3) conclude that $x_n \in \mathcal{S}_n$ has cycle type $[(\frac{n}{2})^2]$ and orbits $\{1, \ldots, \frac{n}{2}\} \,\dot\cup\, \{\frac{n}{2}+1, \ldots, n\}$, on the latter still respecting the ordering introduced in the proof of (3.3).

Let $X_n := \langle x_n \rangle \cong C_{\frac{n}{2}}$. By [7, Thm.III.3.14] we have $X_n \leq P_n^2 = \Phi(P_n)$, where $\Phi(P_n)$ denotes the Frattini subgroup of P_n.

(4.2) Proposition. Let $n = 2^m \geq 8$ and let $Q < P_n$ be a maximal subgroup. Then $\mathrm{res}_Q(D)$ is an indecomposable $F[Q]$-module.

Proof. i) Assume to the contrary that $\mathrm{res}_Q(D)$ is decomposable. As we have $X_n \leq \Phi(P_n) \leq Q$, we first consider $\mathrm{res}_{X_n}(D)$. By (2.4) and (3.5) we have $\mathrm{res}_{Y_n}(D) \cong U_{n-2}$ as $F[Y_n]$-modules. As $x_n = y_n^2$ we have $\dim_F \ker_D(x_n - 1)^j = \dim_F \ker_D(y_n - 1)^{2j} = 2j$, for $1 \leq j \leq \frac{n}{2} - 1$, and $\mathrm{res}_{X_n}(D) \cong U'_{\frac{n}{2}-1} \oplus U'_{\frac{n}{2}-1}$, where U'_i denotes the uniserial $F[X_n]$-module such that $\dim_F(U'_i) = i$, for $1 \leq i \leq \frac{n}{2}$. This implies that $\mathrm{res}_Q(D)$ is the direct sum of two uniserial $F[Q]$-modules, thus $\mathrm{soc}(\mathrm{res}_Q(D)) = \mathrm{soc}(\mathrm{res}_{X_n}(D))$ and $\mathrm{soc}(\mathrm{res}_Q(D)/\mathrm{soc}(\mathrm{res}_Q(D))) = \mathrm{soc}(\mathrm{res}_{X_n}(D)/\mathrm{soc}(\mathrm{res}_{X_n}(D)))$.

Let $s_1 := \sum_{i=1}^{\frac{n}{2}} \gamma_i \in M'$ and $s_2 := \sum_{i=1}^{\frac{n}{4}} \gamma_i + \sum_{i=\frac{n}{2}+1}^{\frac{3n}{4}} \gamma_i \in M'$. By (4.1) we have $s_1(x_n + 1) = s_1(y_{\frac{n}{2}} + 1) = 0$. Furthermore, for $1 \leq i \leq \frac{n}{4}$ we have $\gamma_i(x_n + 1) = \gamma_i(w_{\frac{n}{2}} y_{\frac{n}{4}} + 1) = \gamma_i + \gamma_{i+\frac{n}{4}} \in M$, while for $\frac{n}{2} + 1 \leq i \leq \frac{3n}{4}$ we have $\gamma_i(x_n + 1) = \gamma_i(w_n^{-1} w_{\frac{n}{2}} y_{\frac{n}{4}} w_n + 1) = \gamma_i + \gamma_{i+\frac{n}{4}} \in M$. Hence we have $s_2(x_n + 1) = \gamma^+ \in M''$, and $\{\overline{s_1}, \overline{s_2}\} \subseteq D$ is an F-basis of $\mathrm{soc}(\mathrm{res}_{X_n}(D)) = \mathrm{soc}(\mathrm{res}_Q(D))$. For $g \in Q$ and $i = 1, 2$ we either have $s_i g = s_i \in M$ or $s_i g = s_i + \gamma^+ \in M$.

Therefore both $\{1, \ldots, \frac{n}{2}\} \,\dot\cup\, \{\frac{n}{2} + 1, \ldots, n\}$ and $\{1, \ldots, \frac{n}{4}, \frac{n}{2} + 1, \ldots, \frac{3n}{4}\} \,\dot\cup\, \{\frac{n}{4} + 1, \ldots, \frac{n}{2}, \frac{3n}{4} + 1, \ldots, n\}$ are block systems for the permutation action of Q. Letting $\Gamma_i := \{\frac{in}{4} + 1, \ldots, \frac{(i+1)n}{4}\}$, for $i \in \{0, \ldots, 3\}$, we see that $\coprod_{i=0}^{3} \Gamma_i$ is a block system for Q as well, and we conclude that Q is isomorphic to a 2-subgroup of the wreath product $\mathcal{S}_{\frac{n}{4}} \wr \mathcal{S}_4$, where $\mathcal{S}_{\frac{n}{4}}$ acts on Γ_0 and $\mathcal{S}_4$ acts on $\{\Gamma_0, \ldots, \Gamma_3\}$. Furthermore, if $g \in Q$ fixes one of the Γ_i, then it fixes all of them, implying that Q is isomorphic to a 2-subgroup of $\mathcal{S}_{\frac{n}{4}} \wr V_4$, where $V_4 \trianglelefteq \mathcal{S}_4$. By (3.1) the 2-part of the group order $|\mathcal{S}_{\frac{n}{4}} \wr V_4|$ is given as $|\mathcal{S}_{\frac{n}{4}} \wr V_4|_2 = (2^{\frac{n}{4}-1})^4 \cdot 2^2 = 2^{n-2}$. As we have $|P_n| = 2^{n-1}$ and $|Q| = 2^{n-2}$, we conclude that $Q = P_n \cap (\mathcal{S}_{\frac{n}{4}} \wr V_4)$.

ii) Let $s_3 := \sum_{i=1}^{\frac{n}{8}} \gamma_i + \sum_{i=\frac{n}{4}+1}^{\frac{3n}{8}} \gamma_i + \sum_{i=\frac{n}{2}+1}^{\frac{5n}{8}} \gamma_i + \sum_{i=\frac{3n}{4}+1}^{\frac{7n}{8}} \gamma_i \in M'$, as well as $\overline{s_3} := s_3 + M'' \in M'/M'' \cong D$ and $\overline{\overline{s_3}} := \overline{s_3} + \mathrm{soc}(\mathrm{res}_Q(D)) \in D/\mathrm{soc}(\mathrm{res}_Q(D))$. Hence we have $\overline{\overline{s_3}} \neq 0$. Furthermore, for $1 \leq i \leq \frac{n}{8}$ we have $\gamma_i(x_n + 1) = \gamma_i(w_{\frac{n}{2}} y_{\frac{n}{4}} + 1) = \gamma_i + \gamma_{i+\frac{n}{4}}$, while for $\frac{n}{4} + 1 \leq i \leq \frac{3n}{8}$ we have $\gamma_i(x_n + 1) = \gamma_i(w_{\frac{n}{2}} w_{\frac{n}{4}} y_{\frac{n}{8}} + 1) = \gamma_i + \gamma_{i-\frac{n}{4}} \cdot w_{\frac{n}{4}} y_{\frac{n}{8}} = \gamma_i + \gamma_{i-\frac{n}{8}}$. For $\frac{n}{2} + 1 \leq i \leq \frac{5n}{8}$ and $\frac{3n}{4} + 1 \leq i \leq \frac{7n}{8}$ we argue analogously. Hence we have $s_3(x_n + 1) = s_2$ and $\overline{\overline{s_3}}(x_n + 1) = 0$, and we conclude $\overline{\overline{s_3}} \in \mathrm{soc}(\mathrm{res}_Q(D)/\mathrm{soc}(\mathrm{res}_Q(D)))$, and $\overline{\overline{s_3}}(g + 1) = 0$ for all $g \in Q$.

We have $y_{\frac{n}{4}} \in P_n \cap \mathcal{S}_{\frac{n}{4}} \leq Q$. But $s_3(y_{\frac{n}{4}} + 1) = \sum_{i=1}^{\frac{n}{8}} (\gamma_i + \gamma_i \cdot w_{\frac{n}{4}} y_{\frac{n}{8}}) = \sum_{i=1}^{\frac{n}{4}} \gamma_i \notin \langle \gamma^+, s_1, s_2 \rangle_F$ implies $\overline{\overline{s_3}}(y_{\frac{n}{4}} + 1) \neq 0$, a contradiction. ♯

(4.3) Remark. Note that part i) of the above proof even holds for $n = 2^m \geq 4$, while the assumption $n = 2^m \geq 8$ is needed in part ii) to ensure the existence of $s_3 \in M$, which eventually contradicts the indecomposability of $\mathrm{res}_Q(D)$. For $n = 4$ the analysis in part i) yields $Q = V_4 \trianglelefteq \mathcal{S}_4$, where $\mathrm{res}_Q(D)$ by (1.4)b) and (2.8) indeed is decomposable.

(4.4) Let still $p = 2$, but let now $n \in \mathbb{N}$ be even but not a 2-power. We keep the notation of (3.1) and (3.6). In particular, the 2-adic expansion of n has $l \geq 2$ parts, and the smallest occurring 2-power is $n_l \geq 2$.

Let $\Omega_1 = \Omega_1' \,\dot\cup\, \Omega_1''$, where $\Omega_1' := \{1, \ldots, \frac{n_1}{2}\}$ and $\Omega_1'' := \{\frac{n_1}{2} + 1, \ldots, n_1\}$. Furthermore, the Sylow 2-subgroup $P_{n_1} \leq \mathcal{S}_{\Omega_1}$ is a semidirect product $P_{n_1} = (P_{n_1}' \times P_{n_1}'') \rtimes \langle w_{n_1} \rangle$, where $P_{n_1}' \leq \mathcal{S}_{\Omega_1'}$ and $P_{n_1}'' \leq \mathcal{S}_{\Omega_1''}$ are Sylow 2-subgroups. We have $P_{n_1}' \cong P_{\frac{n_1}{2}} \cong P_{n_1}''$ and $\langle w_{n_1} \rangle \cong C_2$. Let $q_1 : P_{n_1} \to \langle w_{n_1} \rangle \cong C_2$ denote the natural group epimorphism with kernel $\ker(q_1) = P_{n_1}' \times P_{n_1}''$.

As in (4.1) let $x_{n_j} \in P_{n_j} \leq \mathcal{S}_{n_j}$ and $X_{n_j} := \langle x_{n_j} \rangle \leq P_{n_j}$, for $1 \leq j \leq l$, as well as $X_n := \prod_{j=1}^{l} X_{n_j} \leq P_n \leq \mathcal{S}_n$. By [7, Exc.III.3.9] we have $\Phi(P_n) = \prod_{j=1}^{l} \Phi(P_{n_j})$, and $X_{n_j} \leq \Phi(P_n)$, for $1 \leq j \leq l$. Furthermore, by (3.6) we have $\mathrm{res}_{X_n}(M) \cong \bigoplus_{j=1}^{l} \mathrm{res}_{X_{n_j}}(F[Y_{n_j}]) \cong \bigoplus_{j=1}^{l} (F[X_{n_j}] \oplus F[X_{n_j}])$, where again the action on the j-th summand is given by the projection $p_j|_{X_n} : X_n \to X_{n_j}$.

(4.5) Proposition. Let $n \in \mathbb{N}$ be even, but not a 2-power, and let $Q < P_n$ be a maximal subgroup. Then $\mathrm{res}_Q(D)$ does not have a direct $F[Q]$-summand of F-dimension $\frac{\dim_F(D)}{2}$.

Proof. Let $x_{n_j}^+ := (x_{n_j} + 1)^{\frac{n_j}{2} - 1} = \sum_{i=0}^{\frac{n_j}{2} - 1} x_{n_j}^i \in F[X_n]$, for $1 \leq j \leq l$, yielding $x_{n_j}^+ \cdot (x_{n_j} + 1) = 0 \in F[X_n]$. Let $\gamma_{1'}^+ := \sum_{i \in \Omega_1'} \gamma_i \in M'$ and $\gamma_{1''}^+ := \sum_{i \in \Omega_1''} \gamma_i \in M'$, implying $\gamma_{1'}^+ + \gamma_{1''}^+ = \gamma_1^+ \in M'$. Let $u := \gamma_{n_1} + \gamma_{n_1+1} \in M'$ and $v := \gamma_1 + \gamma_{\frac{n_1}{2}+1} \in M'$, hence we have $ux_{n_1}^+ = \gamma_{1'}^+ \notin M''$ and $vx_{n_1}^+ = \gamma_1^+ \notin M''$. As for the action of X_{n_1} on M we have $\dim_F \ker_M(x_{n_1}^+) = n - 2$, we conclude that $\dim_F \ker_{M'}(x_{n_1}^+) \geq \dim_F(M') - 2$ and $\dim_F \ker_D(x_{n_1}^+) \geq \dim_F(D) - 2$. Thus $\dim_F \ker_D(x_{n_1}^+) = \dim_F(D) - 2$ and $Dx_{n_1}^+ = \langle \overline{\gamma_{1'}^+}, \overline{\gamma_{1''}^+} \rangle_F$.

We fix a direct sum decomposition of $\mathrm{res}_Q(D)$. By the above there either is a uniquely determined summand $U \leq \mathrm{res}_Q(D)$ on which $x_{n_1}^+ \in F[X_n]$ does not act by the zero map, or there are exactly two of them, U' and U'' say. In the first case, the projective $F[X_{n_1}]$-module $F[X_{n_1}] \oplus F[X_{n_1}]$ is a direct summand of $\mathrm{res}_{X_{n_1}}(U)$. As we have $n \leq 2n_1 - 2$, we conclude $\dim_F(U) \geq 2 \cdot \dim_F(F[X_{n_1}]) = n_1 \geq \frac{n+2}{2} > \frac{n-2}{2} = \frac{\dim_F(D)}{2}$. We may assume that there are two summands U' and U'' of $\mathrm{res}_Q(D)$ on which $x_{n_1}^+ \in F[X_n]$ does not act by the zero map, and there are $a', a'', b', b'' \in F$ such that $0 \neq u' := a' \cdot \overline{\gamma_{1'}^+} + a'' \cdot \overline{\gamma_{1''}^+} \in U'$ and $0 \neq u'' := b' \cdot \overline{\gamma_{1'}^+} + b'' \cdot \overline{\gamma_{1''}^+} \in U''$; in particular $\{u', u''\}$ is F-linearly independent.

We first consider the case $q_1 p_1(Q) = \langle w_{n_1} \rangle$, see (4.4). There is $g \in Q$ such that $\gamma_{1'}^+ g = \gamma_{1''}^+$ and $\gamma_{1''}^+ g = \gamma_{1'}^+$. If $a' \neq a''$ then using g we conclude that $0 \neq (a' + a'') \cdot \overline{\gamma_1^+} \in U'$, while if $a' = a''$ then we have $\overline{\gamma_1^+} \in U'$ anyway. Similarly we have $\overline{\gamma_1^+} \in U''$ and $\overline{\gamma_1^+} \in U' \cap U''$, a contradiction. We may assume that $q_1 p_1(Q) = \{1\}$. By [7, Thm.III.3.14] we have $|Q| = \frac{|P_n|}{2} = |\ker(q_1 p_1)|$, and $Q = \ker(q_1 p_1) = (P_{n_1}' \times P_{n_1}'') \times \prod_{j=2}^{l} P_{n_j}$. In particular we have $Y_{n_j} \leq Q$ for $2 \leq j \leq l$. Similarly to the proof of (3.7) let $V_j \leq \mathrm{res}_Q(D)$ be the uniquely determined $F[Q]$-summand on which $y_{n_j}^+ \in F[Q]$ does not act by the zero map; we have $\overline{\gamma_j^+} \in V_j$, for $2 \leq j \leq l$. As $\{u', u''\}$ is F-linearly independent there are $a, b \in F$ such that $\overline{\gamma_1^+} = a \cdot u' + b \cdot u'' \in D$. Since $a \cdot u' + b \cdot u'' + \sum_{j=2}^{l} \overline{\gamma_j^+} = \overline{\gamma^+} = 0 \in D$, we conclude $U' = U'' = V_2 = \ldots = V_l$, a contradiction. $\sharp$

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