Definition of Connection Matrix

Attractor-Repeller pair Decomposition of S

$\mathcal{M}(S) = \{M(p) \mid p \in (\{1,2\}, 2 > 1)\}$

Index Triple: $N(\emptyset) \subset N(1) \subset N(12)$

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Short Exact Sequence

$$0 \to C_*(N(1), N(\emptyset)) \xrightarrow{\iota(12,1)} C_*(N(12), N(\emptyset)) \xrightarrow{\rho(2,12)} C_*(N(12), N(1)) \to 0$$

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Long Exact Sequence

$$\xrightarrow{\Delta_{n+1}(1,2)} H_n(N(1), N(\emptyset)) \xrightarrow{\iota_n(12,1)} H_n(N(12), N(\emptyset)) \xrightarrow{\rho_n(2,12)} H_n(N(12), N(1))$$
$$\xrightarrow{\Delta_n(1,2)} H_{n-1}(N(1), N(\emptyset)) \xrightarrow{\iota_{n-1}(12,1)} H_{n-1}(N(12), N(\emptyset)) \xrightarrow{\rho_{n-1}(2,12)} H_{n-1}(N(12), N(1))$$

Long Exact Sequence

$$\xrightarrow{\Delta_{n+1}(1,2)} \begin{array}{c} H_n(N(1), N(\emptyset)) \\ || \\ CH_n(M(1)) \end{array} \xrightarrow{\iota_n(12,1)} \begin{array}{c} H_n(N(12), N(\emptyset)) \\ || \\ CH_n(M(12)) \\ || \\ CH_n(M(S)) \end{array} \xrightarrow{\rho_n(2,12)} \begin{array}{c} H_n(N(12), N(1)) \\ || \\ CH_n(M(2)) \end{array} \xrightarrow{\Delta_n(1,2)}$$

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$$\xrightarrow{\Delta_{n+1}(1,2)} \begin{array}{c} H_n(N(1), N(\emptyset)) \\ || \\ CH_n(M(1)) \end{array} \xrightarrow{\iota_n(12,1)} \begin{array}{c} H_n(N(12), N(\emptyset)) \\ || \\ CH_n(M(12)) \\ || \\ CH_n(M(S)) \end{array} \xrightarrow{\rho_n(2,12)} \begin{array}{c} H_n(N(12), N(1)) \\ || \\ CH_n(M(2)) \end{array} \xrightarrow{\Delta_n(1,2)}$$

This contains ALL the topological information associated with the Conley index for this attractor repeller pair decomposition $M(1), M(2))$ of S. Can we encode this using the Conley indices of the Morse sets?

Long Exact Sequence

$$\xrightarrow{\Delta_{n+1}(1,2)} \underset{\displaystyle CH_n(M(1))}{\underset{||}{H_n(N(1),N(\emptyset))}} \xrightarrow{\iota_n(12,1)} \underset{\displaystyle CH_n(M(S))}{\underset{||}{H_n(N(12),N(\emptyset))}} \xrightarrow{\rho_n(2,12)} \underset{\displaystyle CH_n(M(2))}{\underset{||}{H_n(N(12),N(1))}} \xrightarrow{\Delta_n(1,2)}$$

Connection Matrix:

$$\Delta : \oplus_{p=1}^{2} CH_*(M(p)) \to \oplus_{p=1}^{2} CH_*(M(p))$$

If

$$\Delta := \begin{bmatrix} 0 & \Delta_*(1,2) \\ 0 & 0 \end{bmatrix}$$

then Δ is a boundary operator, i.e. a degree -1 map and $\Delta^2 = 0$ AND

$$CH_*(S) \cong \ker \Delta / \operatorname{im} \Delta = H\Delta_*$$

Morse Decomposition of S

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Morse Decomposition of S

$\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$

$\mathcal{I}$ set of intervals in $\mathcal{P}$

$\mathcal{I}^2$ set of pairs of *adjacent intervals* in $\mathcal{P}$

$$\mathcal{I}^2 := \{(I, J) \in \mathcal{I} \times \mathcal{I} \mid I \cup J \in \mathcal{I},\ I \cap J = \emptyset,\ i \in I, j \in J \Rightarrow i \not> j\}$$

$\mathcal{I}^3$ set of *adjacent triples* of intervals in $\mathcal{P}$

$$\mathcal{I}^3 := \Big\{(I,J,K) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I} \mid (I,J) \in \mathcal{I}^2,\ (J,K) \in \mathcal{I}^2,$$
$$I \cap K = \emptyset,\ i \in I, k \in K \Rightarrow i \not> k\}$$

$\mathcal{I}^3$ set of *adjacent triples* of intervals in $\mathcal{P}$

$$\mathcal{I}^3 := \Big\{(I, J, K) \in \mathcal{I} \times \mathcal{I} \times \mathcal{I} \mid (I, J) \in \mathcal{I}^2,\ (J, K) \in \mathcal{I}^2, \\ I \cap K = \emptyset,\ i \in I, k \in K \Rightarrow i \not> k\}$$

$\mathcal{A}$ set of *attracting intervals* in $\mathcal{P}$

$$\mathcal{A} := \{I \in \mathcal{I} \mid i \in I, i > j \Rightarrow j \in I\}$$

Index Filtration

Attractor-repeller pair decomposition of S:
$\mathcal{M}(S) = \{M(p) \mid p \in (\{1,2\}, 2 > 1)\}$
Index Triple: $N(\emptyset) \subset N(1) \subset N(12)$

Index Filtration $\{N(I) \mid I \in \mathcal{A}\}$ satisfying:

1. for every $I \in \mathcal{A}$, $(N(I), N(\emptyset))$ is an index pair for $M(I)$.

2. for all $I, J \in \mathcal{A}$

$$
\begin{aligned}
N(I) \cap N(J) &= N(I \cap J) \\
N(I) \cup N(J) &= N(I \cup J)
\end{aligned}
$$

Index Filtration

Remark: Given $I \in \mathcal{I}$, there exists $I_A \in \mathcal{A}$ such that $(I_A, I) \in \mathcal{I}^2$ and $I_A \cup I \in \mathcal{A}$.

Propositon: $(N(I_A \cup I), N(I_A))$ is an index pair for $M(I)$

Exact Sequences

A-R pair decomposition of S: $\mathcal{M}(S) = \{M(p) \mid p \in (\{1,2\}, 2 > 1)\}$
Index Triple: $N(\emptyset) \subset N(1) \subset N(12)$
Short Exact Sequence

$$0 \to C_*(N(1), N(\emptyset)) \xrightarrow{\iota(12,1)} C_*(N(12), N(\emptyset)) \xrightarrow{\rho(2,12)} C_*(N(12), N(1)) \to 0$$

Let $(I, J) \in \mathcal{I}^2$

$$0 \to C_*(N(I_A I), N(I_A)) \xrightarrow{\iota(IJ,I)} C_*(N(I_A IJ), N(I_A)) \xrightarrow{\rho(J,IJ)} C_*(N(I_A IJ), N(I_A I)) \to 0$$

Long Exact Sequence

$$\begin{array}{ccccccc} \xrightarrow{\Delta_{n+1}(I,J)} & H_n(N(I_A I), N(I_A)) & \xrightarrow{\iota_n(IJ,I)} & H_n(N(I_A IJ), N(I_A)) & \xrightarrow{\rho_n(J,IJ)} & H_n(N(I_A IJ), N(I_A I)) & \xrightarrow{\Delta_n(I,J)} \\ & \| & & \| & & \| & \\ & CH_n(M(I)) & & CH_n(M(IJ)) & & CH_n(M(J)) & \end{array}$$

Chain Complex Braid

A *chain complex braid* over $(\mathcal{P}, >)$ consists of chain complexes and chain maps satisfying:
1. For each $I \in \mathcal{I}$ there is chain complex $C(I)$
2. For each $(I, J) \in \mathcal{I}^2$ there are chain maps

$$\iota(IJ, I) : C(I) \to C(IJ) \quad \rho(J, IJ) : C(IJ) \to C(J)$$

which satisfy:

1. $0 \to C(I) \xrightarrow{\iota(IJ,I)} C(IJ) \xrightarrow{\rho(J,IJ)} C(J) \to 0$ is exact

2. if $(I, J), (J, I) \in \mathcal{I}^2$ then $\rho(J, IJ)\iota(IJ, I) = \mathrm{id}\,|_{C(I)}$

3. if $(I, J, K) \in \mathcal{I}^3$, then the braid diagram commutes.

Graded Module Braid

A *graded module braid* $\mathcal{H}$ over $(\mathcal{P}, >)$ consists of graded modules and maps satisfying:

1. For each $I \in \mathcal{I}$ there is graded module $H(I)$
2. For each $(I, J) \in \mathcal{I}^2$ there are degree 0 maps

$$\iota(IJ, I) : C(I) \to C(IJ) \quad \rho(J, IJ) : C(IJ) \to C(J)$$

and a degree -1 map

$$\Delta(I, J) : H(J) \to H(I)$$

which satisfy:

1. $\to H(I) \xrightarrow{\iota(IJ,I)} H(IJ) \xrightarrow{\rho(J,IJ)} H(J) \xrightarrow{\Delta(I,J)} H(I) \to$ is exact

2. if $(I,J),(J,I) \in \mathcal{I}^2$ then $\rho(J,IJ)\iota(IJ,I) = \mathrm{id}\,|_{H(I)}$

3. if $(I,J,K) \in \mathcal{I}^3$, then the braid diagram commutes.

A *graded module braid isomorphism* from $\mathcal{H}$ to $\mathcal{G}$ is collection of graded module isomorphisms $\theta(I) : H(I) \to G(I)$, $I \in \mathcal{I}$ such that for every $(I,J) \in \mathcal{I}^2$ the following diagram commutes:

$$\begin{array}{ccccccccc}
\to & H(I) & \overset{\iota(IJ,I)}{\longrightarrow} & H(IJ) & \overset{\rho(J,IJ)}{\longrightarrow} & H(J) & \overset{\Delta(I,J)}{\longrightarrow} & H(I) & \to \\
 & \downarrow\theta(I) & & \downarrow\theta(IJ) & & \downarrow\theta(J) & & \downarrow\theta(I) & \\
\to & G(I) & \overset{\iota(IJ,I)}{\longrightarrow} & G(IJ) & \overset{\rho(J,IJ)}{\longrightarrow} & G(J) & \overset{\Delta(I,J)}{\longrightarrow} & G(I) & \to
\end{array}$$