Rigorous numerics and the Hénon map

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Reference:

"Towards Automated Chaos Verification" with O. Junge and K. Mischaikow, to appear in Proc. Equadiff 2003.

The Lefschetz number on the pointed space $(N_1/N_0, [N_0])$ is

$$L(I, f) := \sum_{n=0}^{\infty} (-1)^n tr f_{N,n}$$

where $I = N_1/N_0$.

Theorem. If $L(I, f) \neq 0$, then f has a fixed point in I.

Similarly,

1. if $L(I, f^k) \neq 0$, then f has a periodic orbit of period k

2. if $L(I, f_{C_n} \circ \ldots \circ f_{C_1}) \neq 0$, then f has a periodic orbit traveling through components $C_1, \ldots, C_n$

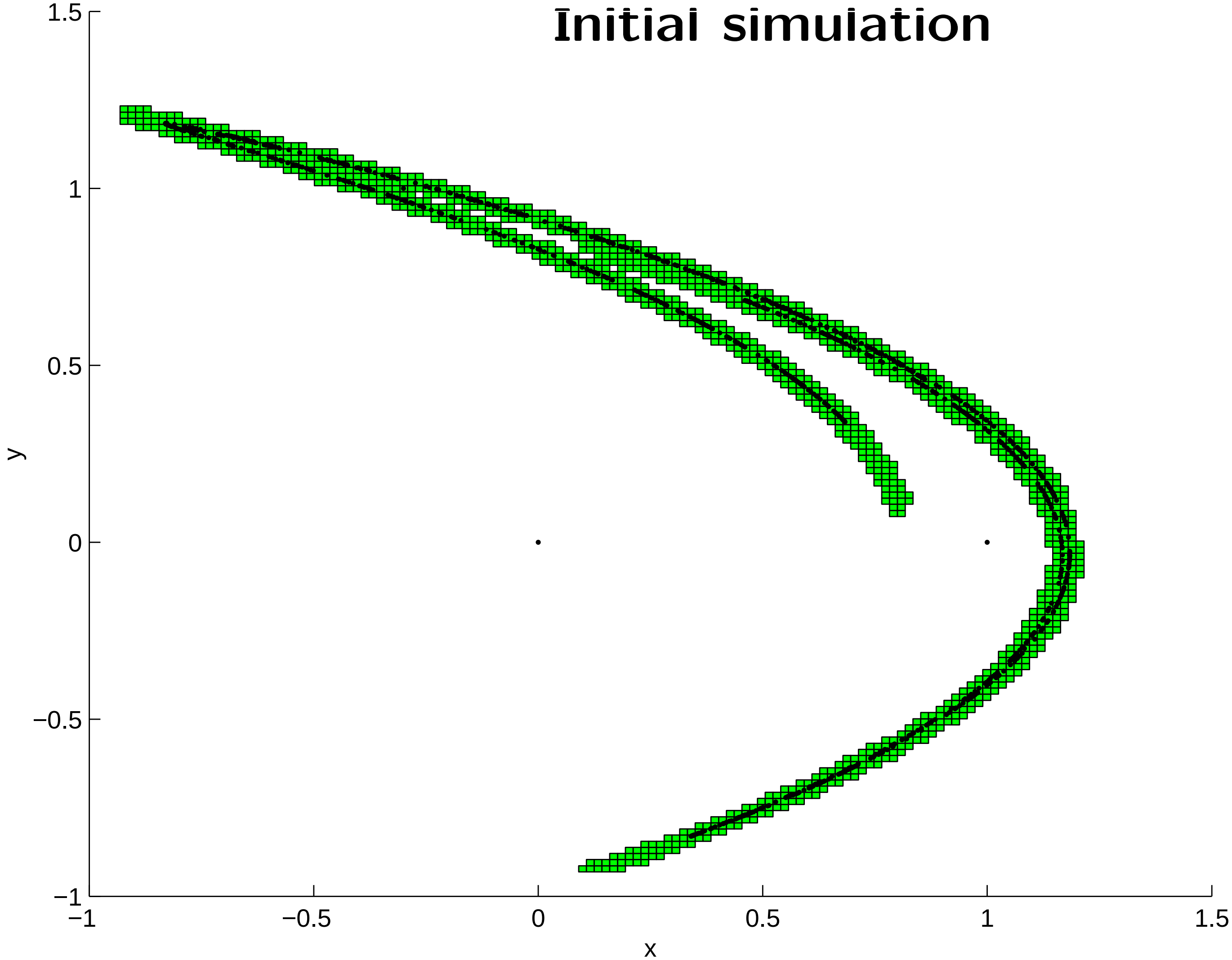
Example 1: the Hénon map

$$h : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 - ax^2 + y/5 \\ 5bx \end{pmatrix}$$

With parameter values $a = 1.3$ and $b = 0.2$.

Initial simulation



Finite representation

From these simulations, we choose a box to initialize a tree in GAIO.

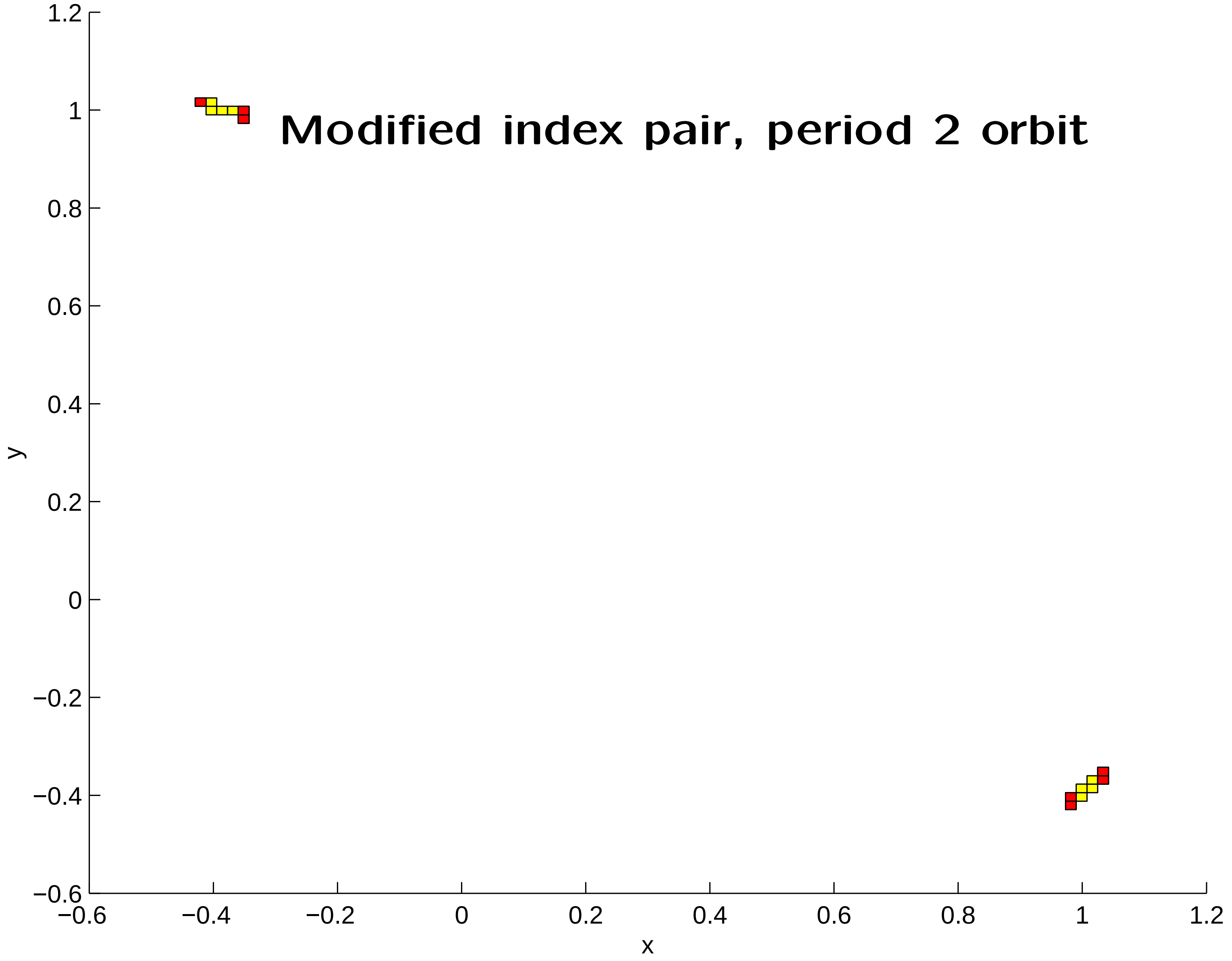
For our computations we choose to initially subdivide each box 14 times (subdividing 7 times in each of the two directions). This results in 1188 boxes, each of box radius $[0.0087, 0.0087]$ covering the maximal invariant set and a transition matrix, $M_{\mathcal{H}}$, on these boxes.

Period 2 orbit

- Initial guess: nonzero entries of the diagonal of $M^2_{\mathcal{H}}$ (boxes 553 and 78)

- "grow" this initial two box collection into a combinatorial isolated invariant set $\mathcal{S}$ for $\mathcal{H}$

- construct the corresponding modified combinatorial index pair, $[\mathcal{P}_1, \mathcal{P}_2]$


Modified index pair, period 2 orbit
y
x
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2

Period 2 orbit, index

Compute the index

- construct auxiliary pair $[\mathcal{Q}_1, \mathcal{Q}_2]$
- use CHomP, with $[\mathcal{P}_1, \mathcal{P}_2]$, $[\mathcal{Q}_1, \mathcal{Q}_2]$, and $M_{\mathcal{H}}$ on $\mathcal{P}_1$

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^2,\ 0,\ 0,\ \ldots).$$

For an appropriate choice of basis,

$$h_{P,1} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Each generator of $H_1(|\mathcal{P}_1|, |\mathcal{P}_0|)$ lies in a distinct connected component of $|\mathcal{P}_1| \setminus |\mathcal{P}_0|$.

Period 2 orbit, proof

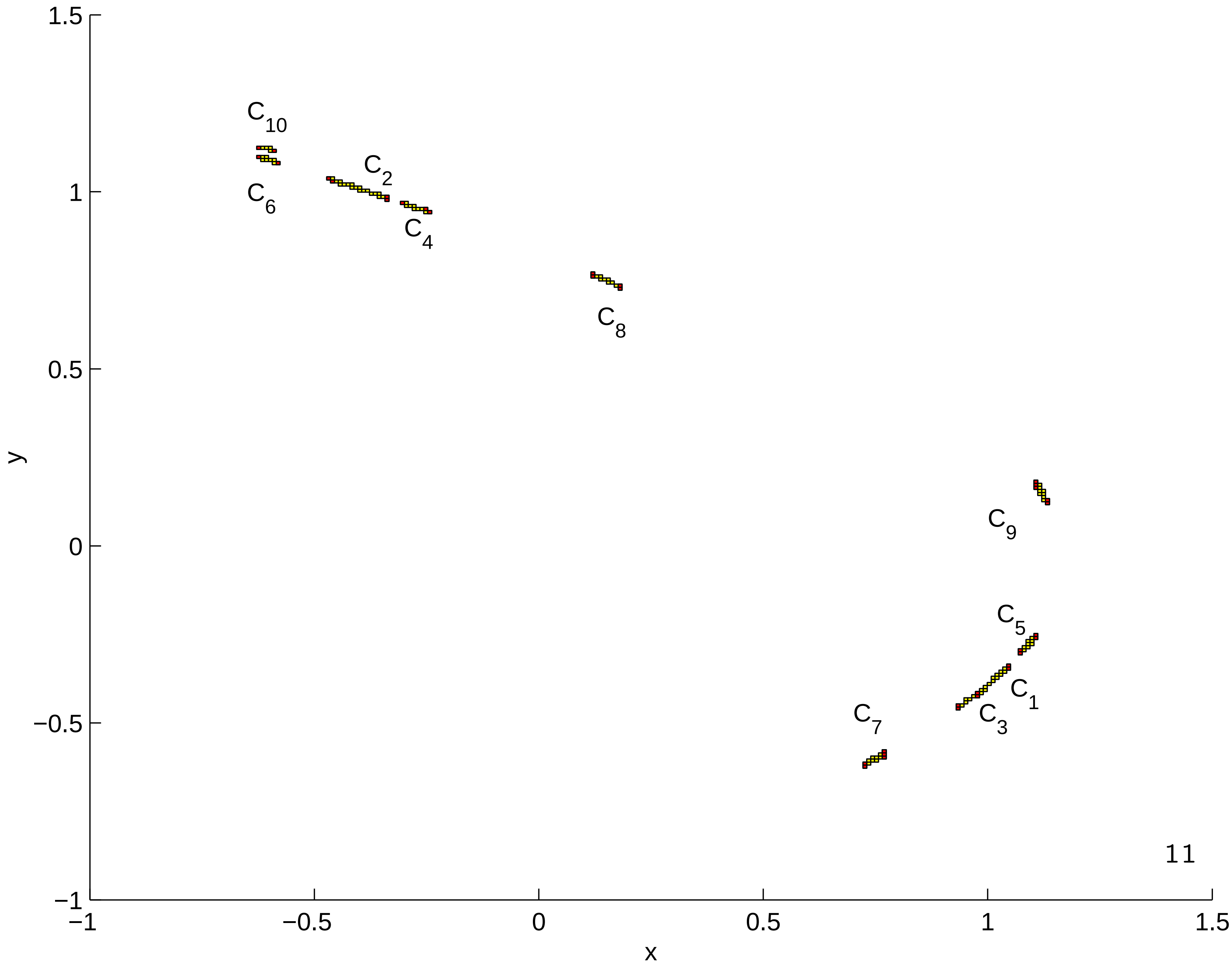
Theorem. There exists a periodic orbit of minimal period two for the Hénon map with elements in each of the distinct connected components of $\mathcal{S}$.

Proof. The Lefschetz number $L(\mathcal{I}, h^2) = -tr(h^2_{P,1}) = -2$ is nonzero so $\mathcal{S}$ contains a periodic point of period two. Furthermore, this orbit has minimal period two since the transition graph on the two connected components of $\mathcal{S}$ prohibits there being a fixed point.

A connecting orbit

Initial guess: an orbit connecting a period 2 orbit to a period 4 orbit in $M_{\mathcal{H}}$ at depth 16

Grow isolating neighborhood, construct modified index pair.


C10
C6
C2
C4
C8
C9
C5
C1
C3
C7
y
x
1.5
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
1.5

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^{10},\ 0,\ 0,\ \ldots).$$

$$h_{P,1} := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Each of the 10 generators lies in a distinct component.

Connecting orbit, proof

Theorem. There exists an orbit for the Hénon map which limits in forwards time to a box neighborhood of a period 4 orbit, and in backwards time to a box neighborhood containing a period 2 orbit.

Proof. The periodic orbits exist:

$$\operatorname{tr} h^2_{C_1 \cup C_2, 1} = -2 \text{ and } \operatorname{tr} h^4_{C_7 \cup \ldots \cup C_{10}, 1} = -4.$$

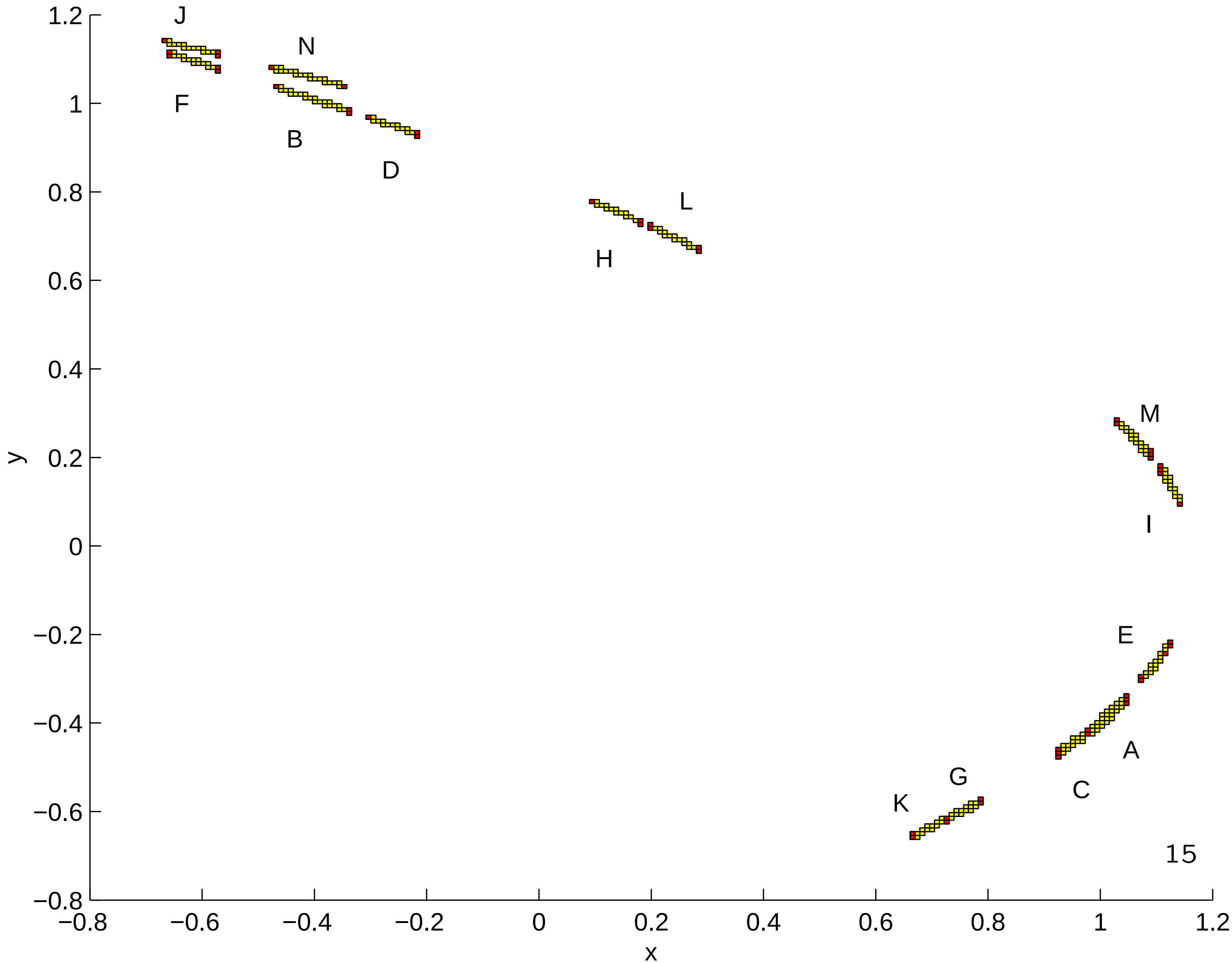
h_{P*} is not shift equivalent to $h_{C_1 \cup C_2 *} \oplus h_{\cup_{i \in \{7,8,9,10\}} C_i *}$, so the Conley indices of $\text{Inv}(|\mathcal{S}|, h)$ and $\text{Inv}(\cup_{i \in \{1,2,7,8,9,10\}} C_i, h)$ are different. Hence $\text{Inv}(|\mathcal{S}|, h) \neq \text{Inv}(\cup_{i \in \{1,2,7,8,9,10\}} C_i, h)$

This, in addition to the transition information given by $M_{\mathcal{H}}$, completes the proof.

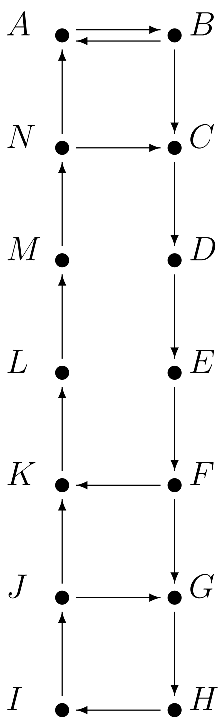
Chaotic symbolic dynamics, 1d-unstable

Initial guess: a period four orbit, a period two orbit, and two connecting orbits in the transition graph given by $M_{\mathcal{H}}$ at depth 16

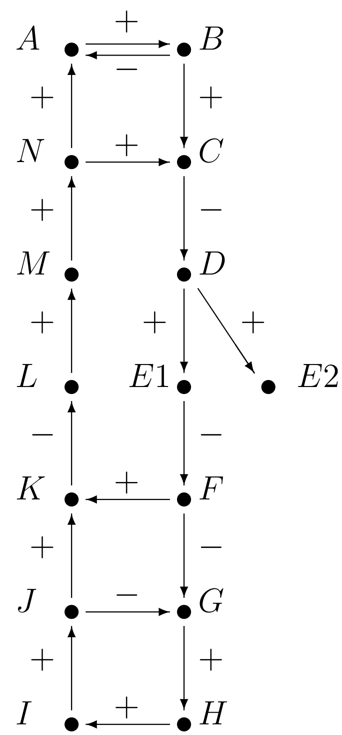
Grow isolating neighborhood, construct modified index pair.


J
N
F
B
D
L
H
M
I
E
A
G
C
K
y
x
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^{15},\ 0,\ 0,\ \ldots).$$



transition graph on components



index map on generators

Theorem There is a set contained in $|\mathcal{S}|$, on which h is semi-conjugate to the symbol subshift given by the transition graph.

Proof.

$$\begin{array}{ccc} S & \xrightarrow{h} & S \\ \rho \downarrow & & \downarrow \rho \\ \Sigma_T & \xrightarrow{\sigma} & \Sigma_T \end{array}$$

$\Sigma_T = \{(Z_i)_i | (Z_i, Z_{i-1})\text{an edge in } T\}$

$\rho(x) = \{Z_i | h^i(x) \in Z_i\}$

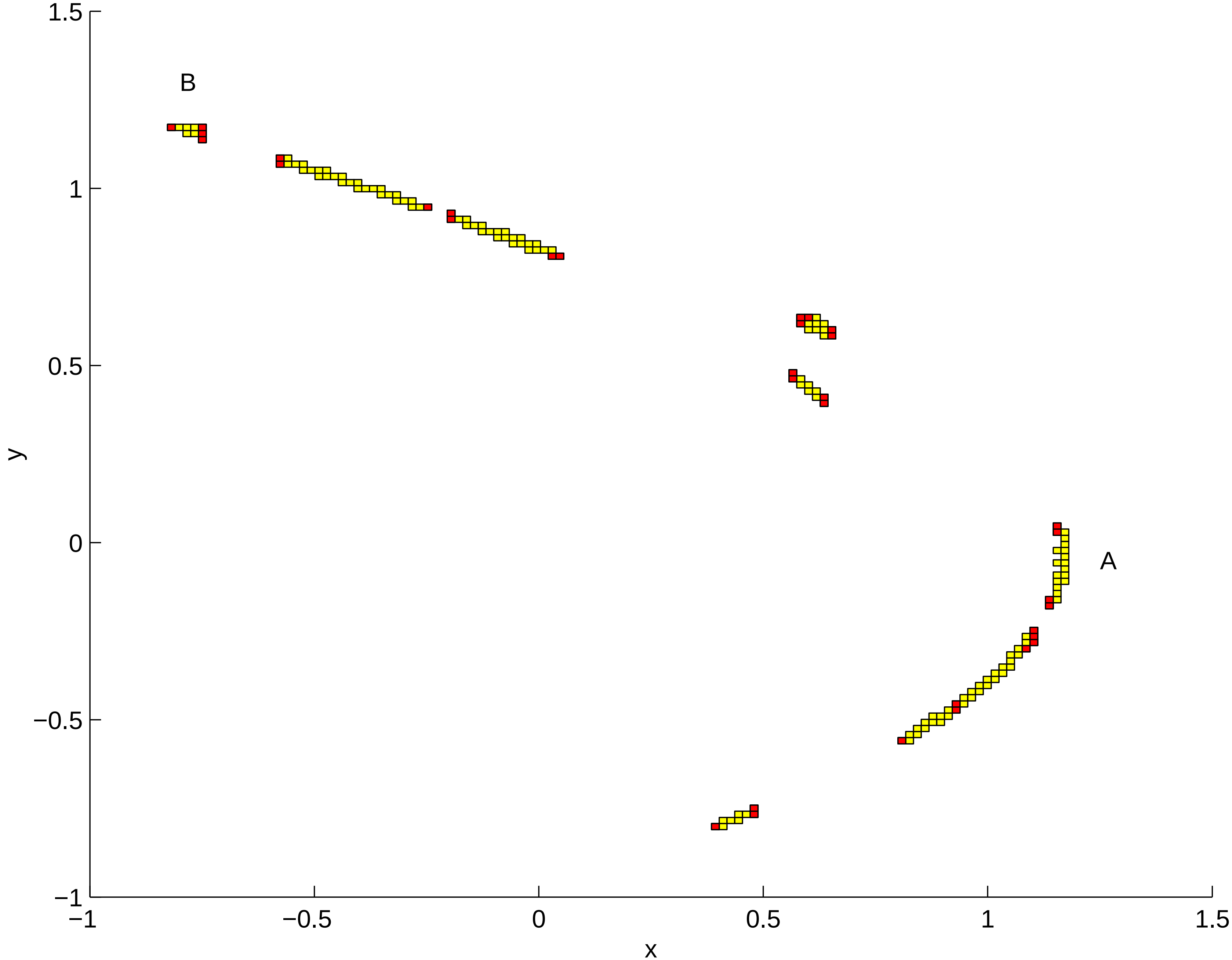
σ is left shift by one symbol

- Study e.g. $\Lambda(I, h^{42}_{JIHG} h_F \dots h_D \dots h_K)$. Since for each periodic symbol sequence, the corresponding Lefschetz number is nonzero, there exists at least one corresponding periodic orbit in S.

- ρ is continuous and S is compact. Therefore, ρ maps onto Σ_T, the closure of periodic orbits.

Transition graph vs the index

Initial guess: connecting orbit from a period two orbit to a fixed point given by $M_{\mathcal{H}}$ at depth 14



1.5
1
0.5
0
−0.5
−1
y
−1
−0.5
0
0.5
1
1.5
x
A
B

The computed index is

$$H_*(|\mathcal{P}_1|, |\mathcal{P}_0|) \cong (0,\ \mathbb{Z}^9,\ 0,\ 0,\ \ldots)$$

$$h_{P,1} := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & {\color{red}0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

$h_{P,1}$ folds the only generator in Component A (column 5) and maps it trivially to the generator in Component B (row 6).

Subdividing $\mathcal{S}$ at depth 14 to a depth of 24, $M_{\mathcal{H}}$ prohibits a connecting orbit of this type.