Why are we learning/developing this mathematics?

Two major scientific developments of the past 50 years:

- Complexity of nonlinear dynamical systems
- Development of information technologies

Structural stability is not generic

Homoclinic tangencies

- Newhouse Phenomenon - strange attractors and arbitrarily long stable periodic orbits intertwined in parameter space.

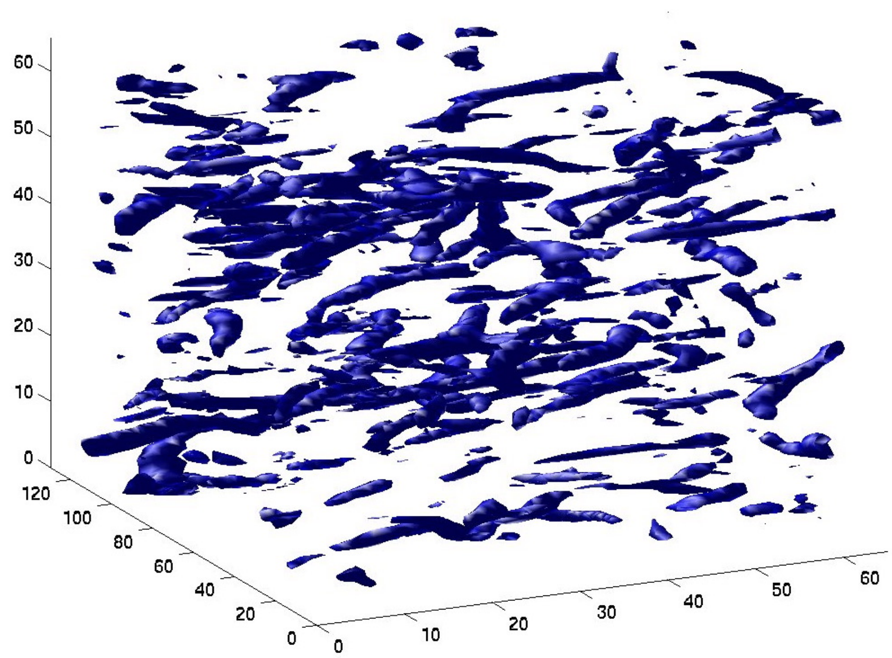
Structures or patterns at multiple scales

- Chaos

- Continuously changing patterns

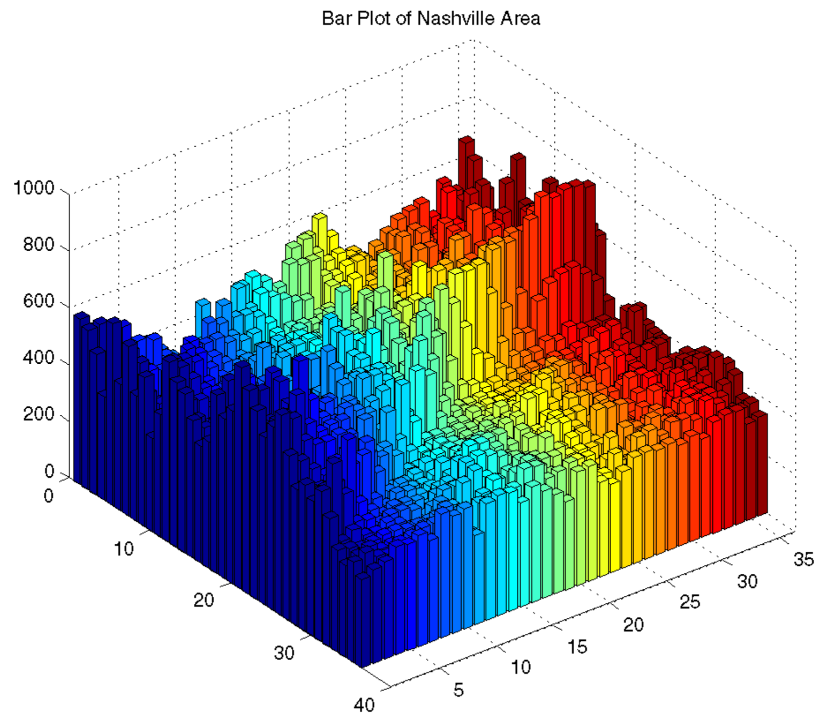
Structures or patterns at multiple scales

- Turbulence - spatial-temporal chaos



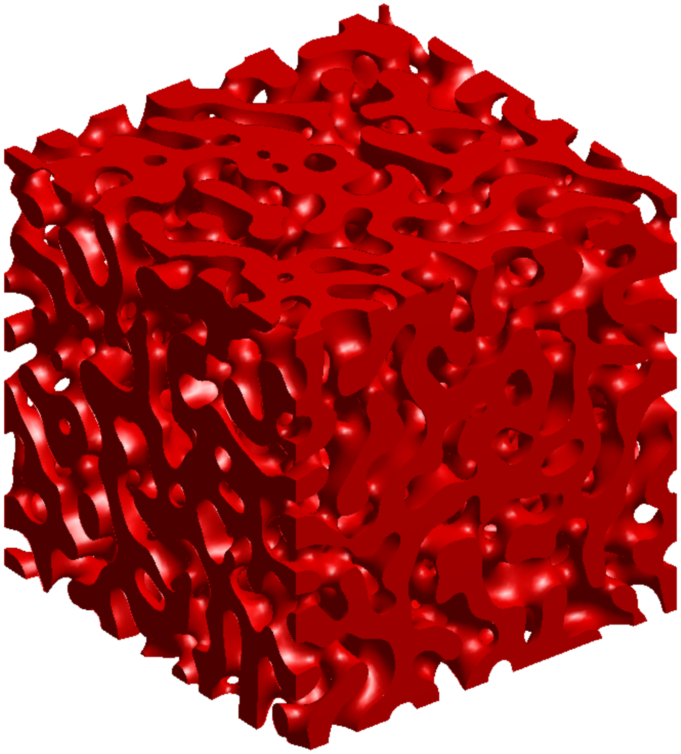
Structures or patterns at multiple scales

- Fractals - topography



Structures or patterns at multiple scales

- Microstructures in materials - transient complexity



We need rigorous mathematical techinques that can do two things:

- Capture those structures that persist under perturbation.

- Identify structures down to a particular scale.

Conley theory (topological generalization of Morse theory) provides us with appropriate tools.

Conley theory (topological generalization of Morse theory) provides us with appropriate tools.

I. Structure Theorems

- Existence of periodic orbits

Conley theory (topological generalization of Morse theory) provides us with appropriate tools.

1. Structure Theorems

- Existence of periodic orbits

- Structure of connecting orbits

Conley theory (topological generalization of Morse theory) provides us with appropriate tools.

1. Structure Theorems

- Existence of periodic orbits

- Structure of connecting orbits

- Existence of symbolic dynamics

Conley theory (topological generalization of Morse theory) provides us with appropriate tools.

1. Structure Theorems

- Existence of periodic orbits
- Structure of connecting orbits
- Existence of symbolic dynamics
- Existence of homoclinic tangencies

Conley theory (topological generalization of Morse theory) provides us with appropriate tools.

1. Structure Theorems

- Existence of periodic orbits
- Structure of connecting orbits
- Existence of symbolic dynamics
- Existence of homoclinic tangencies
- Singular perturbations

Conley theory (topological generalization of Morse theory) provides us with appropriate tools.

1. Structure Theorems

- Existence of periodic orbits
- Structure of connecting orbits
- Existence of symbolic dynamics
- Existence of homoclinic tangencies
- Singular perturbations
- Time series

2. Bifurcation Theorems

- Homoclinic and heteroclinic orbits

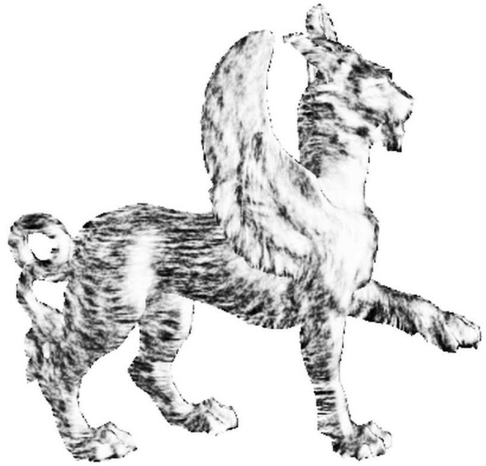
2. Bifurcation Theorems

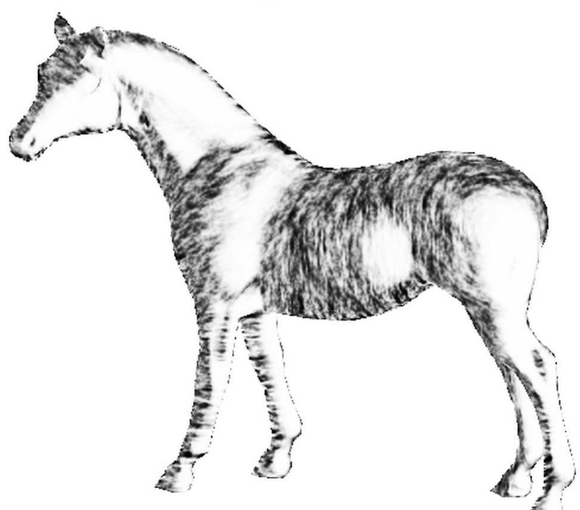
- Homoclinic and heteroclinic orbits

- Resolution or Data Compression

2. Bifurcation Theorems

- Homoclinic and heteroclinic orbits
- Resolution or Data Compression
- Computer Graphics





Limiting factors in applying the Conley index

1. Finding index pairs

Limiting factors in applying the Conley index

1. Finding index pairs

2. Computing the Conley index

Computer Assisted Proof in Dynamics

1. Chaotic dynamics in Lorenz equations

Computer Assisted Proof in Dynamics

1. Chaotic dynamics in Lorenz equations
2. Homoclinic tangencies in Henon

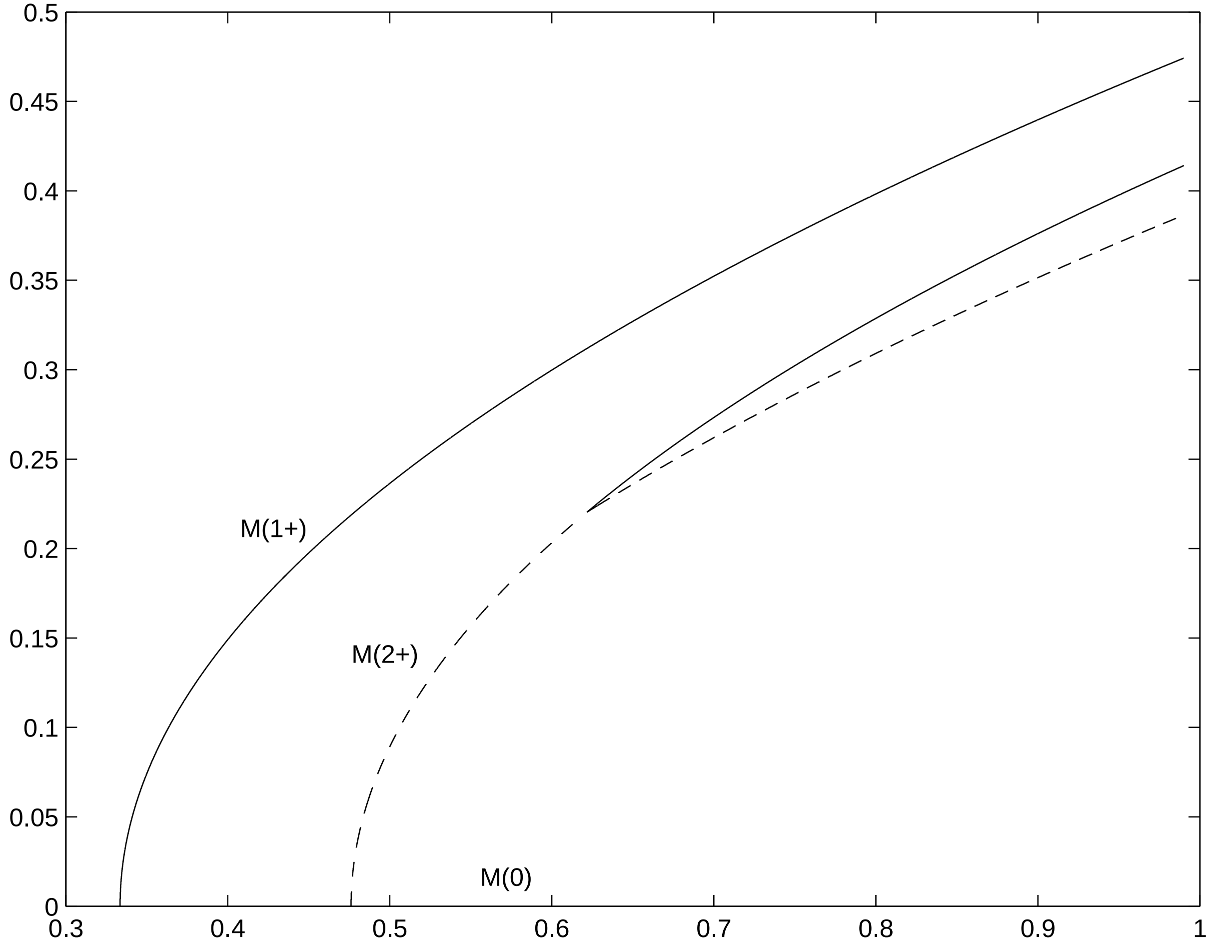
Computer Assisted Proof in Dynamics

1. Chaotic dynamics in Lorenz equations

2. Homoclinic tangencies in Henon

3. Infinite Dimensional Systems

- Bifurcation diagrams (Swift-Hohenberg, Kuramoto-Sivashinsky, FitzHugh-Nagumo)



$$u_t = E(u) = \left\{ \nu - \left(1 + \frac{\partial^2}{\partial x^2} \right)^2 \right\} u - u^3, \qquad u(\cdot, t) \in L^2\left(0, \frac{2\pi}{L}\right),$$

$$u(x,t) = u\left(x + \frac{2\pi}{L}, t\right), \qquad u(-x,t) = u(x,t), \qquad \nu > 0, \qquad (1)$$

Computer Assisted Proof in Dynamics

1. Chaotic dynamics in Lorenz equations

2. Homoclinic tangencies in Henon

3. Infinite Dimensional Systems

- Bifurcation diagrams (Swift-Hohenberg, Kuramoto-Sivashinsky, FitzHugh-Nagumo)

- Global attractors for gradient systems (Swift-Hohenberg)

Computer Assisted Proof in Dynamics

1. Chaotic dynamics in Lorenz equations

2. Homoclinic tangencies in Henon

3. Infinite Dimensional Systems

- Bifurcation diagrams (Swift-Hohenberg, Kuramoto-Sivashinsky, FitzHugh-Nagumo)

- Global attractors for gradient systems (Swift-Hohenberg)

- Chaotic dynamics (Kot-Schaffer)

Computer Assisted Proof in Dynamics

1. Chaotic dynamics in Lorenz equations

2. Homoclinic tangencies in Henon

3. Infinite Dimensional Systems

- Bifurcation diagrams (Swift-Hohenberg, Kuramoto-Sivashinsky, FitzHugh-Nagumo)

- Global attractors for gradient systems (Swift-Hohenberg)

- Chaotic dynamics (Kot-Schaffer)

4. Geometric approximations of vector fields.

Conley Index

$f : X \to X$ a continuous function on a locally compact metric space

$S \subset X$ is *invariant* if for every $x \in S$ there exists a full trajectory $\gamma_x : \mathbf{Z} \to S$ such that

$$\gamma_x(0) = x, \quad \text{and} \quad \gamma_x(n+1) = f(x).$$

$N \subset X$ is an *isolating neighborhood* if

$$\text{Inv}(\text{cl}(N), f) := \{x \in N \mid \exists\, \gamma_x : \mathbf{Z} \to N\} \subset \text{int}(N)$$

Let $P = (P_1, P_0)$ with $P_0 \subset P_1$ be a pair of sets in X. Define

$$f_P : P_1/P_0 \to P_1/P_0$$

by

$$f_P(x) = \begin{cases} f(x) & \text{if } x, f(x) \in P_1 \setminus P_0 \\ [P_0] & \text{otherwise.} \end{cases}$$

Let $P = (P_1, P_0)$ with $P_0 \subset P_1$ be a pair of sets in X. Define

$$f_P : P_1/P_0 \to P_1/P_0$$

by

$$f_P(x) = \begin{cases} f(x) & \text{if } x, f(x) \in P_1 \setminus P_0 \\ [P_0] & \text{otherwise.} \end{cases}$$

A pair of compact sets $P = (P_1, P_0)$ is an *index pair* for f if

- $P_1 \setminus P_0$ is an isolating neighborhood

- f_P is continuous.

In order for f_P to be continuous, P_0 must be an *exit set* for P_1.

Theorem: (Ważewski Principle) If f_P is not homotopically trivial, then

$$\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), f) \neq \emptyset.$$

Theorem: (Ważewski Principle) If f_P is not homotopically trivial, then

$$\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), f) \neq \emptyset.$$

Remark: Homotopy theory is hard to compute with.

Theorem: (Ważewski Principle) If f_P is not homotopically trivial, then

$$\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), f) \neq \emptyset.$$

Remark: Homotopy theory is hard to compute with.

Theorem: (Ważewski Principle) If

$$f_{P*} : H_*(P_1/P_0, [P_0]) \to H_*(P_1/P_0, [P_0])$$

is not nilpotent, then

$$\mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_0), f) \neq \emptyset.$$

Shift Equivalence

The *homology Conley index* of $\text{Inv}(\text{cl}(P_1 \setminus P_0), f)$ is the shift equivalence class of $f_{P*} : H_*(P_1/P_0, [P_0]) \to H_*(P_1/P_0, [P_0])$.

Two group homomorphisms $f : X \to X$ and $g : Y \to Y$ are *shift equivalent* if there exist group homomorphisms $r : X \to Y$ and $s : Y \to X$ and a natural number m such that

$$r \circ f = g \circ r, \quad s \circ g = f \circ s, \quad r \circ s = g^m, \quad s \circ r = f^m$$

Prop: Let $f : X \to X$ and $g : Y \to Y$ be group homomorphisms that are shift equivalent. Then f is nilpotent if and only if g is nilpotent.

Proof: Assume f is not nilpotent. Since $s \circ r = f^m$, neither r nor s are trivial. Assume that $g^k = 0$. Then

$$\begin{aligned} r \circ f &= g \circ r, \\ s \circ r \circ f \circ (s \circ r)^k &= s \circ g \circ r \circ (s \circ r)^k, \\ f^m \circ f \circ (f^m)^k &= s \circ g \circ (r \circ s)^k \circ r, \\ f^{m(k+1)+1} &= s \circ g \circ (g^m)^k \circ r, \\ f^{m(k+1)+1} &= 0. \end{aligned}$$

This contradicts the assumption that f is not nilpotent.

Exercise: $f(x) = 2x$ is not shift equivalent to $g(x) = x$.