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Control Theory – Tutorial 10

To be handed in till: Monday, Jul. 06
To be discussed on: Tuesday, Jul. 07

Exercise 1[2+3] Let $\mathcal{D}$ be a commutative ring (with 1) and let $R_i \in \mathcal{D}^{g_i \times q}$ for $i = 1, 2$ be two $\mathcal{D}$-matrices with the same number of columns. One calls $R \in \mathcal{D}^{g \times q}$ a *common left multiple* of R_1, R_2 if there exist $\mathcal{D}$-matrices A, B such that

$$R = AR_1 = BR_2,$$

and a *least* common left multiple of R_1, R_2 if additionally, any other common left multiple of R_1, R_2 is a left multiple of R, that is,

$$\tilde{R} = \tilde{A}R_1 = \tilde{B}R_2 \quad \Rightarrow \quad \exists X \in \mathcal{D}^{\tilde{g} \times g} : \tilde{R} = XR.$$

1. Show that if $\mathcal{D}$ is Noetherian, then any two matrices (with the same number of columns) possess a least common left multiple.

   Hint: Consider the left kernel of
   $$\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \\ I_q & I_q \end{bmatrix},$$
   which is finitely generated by assumption, and thus generated by the rows of some $\mathcal{D}$-matrix $[A, B, -C]$ ...

2. Show that R is a least common left multiple of R_1, R_2 if and only if $\mathcal{D}^{1 \times g_1} R_1 \cap \mathcal{D}^{1 \times g_2} R_2 = \mathcal{D}^{1 \times g} R$. How non-unique are least common left multiples?

Remark: The fact that any two matrices have a least common left multiple (l.c.l.m.) does not imply, of course, that any two *elements* $r_1, r_2 \in \mathcal{D}$ possess a least common multiple in $\mathcal{D}$, since $\mathcal{D}r_1 \cap \mathcal{D}r_2$ is not necessarily a principal ideal. To avoid confusion, the term l.c.l.m. is usually reserved for the case where $\mathcal{D}$ is a principal ideal ring.

Exercise 2[2+5]
Now let $\mathcal{D} = \mathbb{R}[s]$, $R_i \in \mathcal{D}^{g_i \times q}$, and $\mathcal{B}_i = \{w \in \mathcal{A}^q \mid R_i(\frac{d}{dt})w = 0\}$.

1. Show that $\mathcal{B}_1 + \mathcal{B}_2$ is the system represented by "the" (i.e., any) least common left multiple of R_1, R_2.

   Hint: Write $\mathcal{B}_1 + \mathcal{B}_2 =$

   $$\left\{ w \in \mathcal{A}^q \mid \exists w_1, w_2 \in \mathcal{A}^q : \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \\ I_q & I_q \end{bmatrix} (\tfrac{d}{dt}) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_q \end{bmatrix} w \right\}$$

   and use the elimination of latent variables.

2. Let $\mathcal{K} = \mathbb{R}(s)$ denote the quotient field of $\mathcal{D}$. Recall that the rank of a $\mathcal{D}$-matrix is defined as the rank over $\mathcal{K}$. Show that the following are equivalent:

   (a) $\operatorname{rank}(R_1) + \operatorname{rank}(R_2) = \operatorname{rank} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$.

   (b) $\mathcal{K}^{1 \times g_1} R_1 \cap \mathcal{K}^{1 \times g_2} R_2 = \{0\}$.

   (c) $\mathcal{D}^{1 \times g_1} R_1 \cap \mathcal{D}^{1 \times g_2} R_2 = \{0\}$.

   (d) Any common left multiple of R_1, R_2 is zero.

   (e) $\mathcal{B}_1 + \mathcal{B}_2 = \mathcal{A}^q$.

   Hint: For "(e) $\Rightarrow$?", use the concept of input-dimension.

Exercise 3[8] Generalize the necessary/sufficient conditions for controllability of a series and a parallel connection from Exercise 1 of Tutorial 8 to the case of input-output relations $P_i(\frac{d}{dt})y_i = Q_i(\frac{d}{dt})u_i$ for $i = 1, 2$.