Eva Zerz
Kristina Schindelar SS 2009

Control Theory – Tutorial 11

To be handed in till: Monday, July 13
To be discussed on: Tuesday, July 14

Exercise 1[3] Consider an observable system $x(t+1) = Ax(t)$, $y(t) = Cx(t)$, $x(0) = x_0$. Clearly,

$$\underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{=:O} x_0 = \underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix}}_{=:z},$$

and thus, $x_0 = (O^T O)^{-1} O^T z$. In practice however, z will be subject to disturbance (recall that it usually represents a measurement vector), say, we measure some $\hat{z} = [\hat{y}(0)^T, \ldots, \hat{y}(n-1)^T]^T$, and then the linear system $Ox_0 = \hat{z}$ will not have a solution, in general. Show that it still makes sense to compute

$$\hat{x}_0 := (O^T O)^{-1} O^T \hat{z},$$

as it is the "best estimate" for the true initial state x_0 in the sense that for all $x_0 \in \mathbb{R}^n$, we have

$$\|O\hat{x}_0 - \hat{z}\| \leq \|Ox_0 - \hat{z}\|$$

with equality iff $\hat{x}_0 = x_0$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{np}$. Thus, the output vector arising from starting in $\hat{x}_0$ is the best approximation of $\hat{z}$.

Exercise 2[7] Now consider an observable system $\dot{x} = Ax$, $y = Cx$, $x(0) = x_0$. Let $\varepsilon > 0$ be given. Define the observability operator

$$L : \mathbb{R}^n \to \mathcal{Y} := \mathcal{L}^2([0,\varepsilon], \mathbb{R}^p), \quad x_0 \mapsto \left\{ \begin{array}{ccc} [0,\varepsilon] & \to & \mathbb{R}^p \\ t & \mapsto & Ce^{At}x_0, \end{array} \right.$$

which is linear and bounded (since $\|Lx_0\|_{\mathcal{Y}}^2 := \int_0^\varepsilon \|(Lx_0)(t)\|^2 dt = \ldots$). Observability amounts to the injectivity of L. Show that

$$L^* : \mathcal{Y} \to \mathbb{R}^n, \quad z(\cdot) \mapsto \int_0^\varepsilon e^{A^T t} C^T z(t) dt$$

is the adjoint of L, that is, for all $x_0 \in \mathbb{R}^n$, $z(\cdot) \in \mathcal{Y}$,

$$\langle Lx_0, z(\cdot) \rangle_{\mathcal{Y}} = \langle x_0, L^* z(\cdot) \rangle_{\mathbb{R}^n}.$$

Moreover, $L^*L = W(\varepsilon)$, the observability Gramian. Conclude that $Lx_0 = y(\cdot)$ implies

$$x_0 = W(\varepsilon)^{-1}L^*y(\cdot).$$

Again, when we measure $\hat{y}(\cdot)$ instead of $y(\cdot)$, we set

$$\hat{x}_0 := W(\varepsilon)^{-1}L^*\hat{y}(\cdot).$$

Then we have for all $x_0 \in \mathbb{R}^n$:

$$\|L\hat{x}_0 - \hat{y}\|_{\mathcal{Y}} \leq \|Lx_0 - \hat{y}\|_{\mathcal{Y}}$$

with equality iff $\hat{x}_0 = x_0$.

Exercise 3[2+4+4] Let $P := W_c(\varepsilon)$ and $Q := W_o(\varepsilon)$ be the controllability and observability Gramians of the controllable and observable system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$. Recall that $x_0^T P^{-1} x_0$ is the input energy needed for steering the system from 0 to x_0 in time $\varepsilon > 0$. Similarly, $x_0^T Q x_0$ is the output energy (when $u \equiv 0$ and $x(0) = x_0$) on $[0, \varepsilon]$.

A state x_0 is "good" w.r.t. controllability issues if $x_0^T P^{-1} x_0$ is "small", and "bad" w.r.t. observability questions if $x_0^T Q x_0$ is "small" (an initial state producing a low energy output is hard to distinguish from zero). It is desirable to have a system representation in which any state $e_i \in \mathbb{R}^n$ is either "good" w.r.t. both criteria, or "bad" w.r.t. both criteria, thus making it easy to decide about the "quality" of a state component. For this, we wish to transform (A, B, C, D) into $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = (T^{-1}AT, T^{-1}B, CT, D)$ such that the Gramians of the new system are both diagonal and equal to each other. This procedure is called *balancing*.

1. Show that $\tilde{P}, \tilde{Q}$ can be expressed in terms of P, Q via the congruence relations
$$\tilde{P} = T^{-1}PT^{-T} \quad \text{and} \quad \tilde{Q} = T^T Q T.$$
Conclude that PQ and $\tilde{P}\tilde{Q}$ are similar.

2. Show that the eigenvalues of PQ are real and positive.

3. Show that there exists an invertible matrix T such that
$$\tilde{P} = \tilde{Q} = \operatorname{diag}(\sigma_1, \ldots, \sigma_n),$$
where $\sigma_i > 0$ are the square roots of the eigenvalues of PQ. The σ_i are called the *Hankel singular values* of the system (at time ε).

Hints: Decompose $P = RR^T$; set $S := R^T QR$; then $S > 0$ and $\text{spec}(S) = \text{spec}(PQ)$; let $U^T SU = \Lambda$, U orthogonal, Λ diagonal. Set $T := RU\Lambda^{-1/4}$.

How would you define the quality (as explained above) of a state $e_i \in \mathbb{R}^n$ in the new model?