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SS 2009

Control Theory – Tutorial 12

To be handed in till: Monday, July 20
To be discussed on: Tuesday, July 21

Exercise 1[1+3+1+2+2] Z-transform: For a sequence $f = (f(0), f(1), f(2), \ldots)$, the z-transform is given by $(\mathcal{Z}f)(z) := \sum_{t=0}^{\infty} f(t)z^{-t}$. This is a formal power series in z^{-1}.

1. Prove that $(\mathcal{Z}(\sigma f))(z) = z((\mathcal{Z}f)(z) - f(0))$, where $(\sigma f)(t) = f(t+1)$ for all $t \in \mathbb{N}$.
2. Show that for any matrix $A \in \mathbb{R}^{n\times n}$, there exists $(zI - A)^{-1} \in \mathbb{R}(z)^{n\times n}$, and formally, we have $(zI - A)^{-1} = z^{-1}\sum_{t=0}^{\infty} A^t z^{-t}$. For which $z \in \mathbb{C}$ is this formal power series convergent?
3. Consider a discrete state space system $\sigma x = Ax + Bu$, $y = Cx + Du$ with zero initial state, that is, $x(0) = 0$. Apply the z-transform to both equations and eliminate $\mathcal{Z}x$, that is, find a relation between $\mathcal{Z}y$ and $\mathcal{Z}u$.
4. Let $m = 1$ and set $u = \delta$, where $\delta(t) = \delta_{0t}$ (Kronecker). Check that
$$h = (D, CB, CAB, CA^2B, \ldots)$$
is the output sequence of the discrete state space system with $x(0) = 0$, and compute its z-transform.
5. Show that $\mathcal{Z}(g * u) = \mathcal{Z}g \cdot \mathcal{Z}u$, where $(g * u)(t) = \sum_{i=0}^{t} g(t-i)u(i)$.

Remark: In a discrete state space system with $x(0) = 0$, we have $y = h * u$, where h is the "impulse response" from 4., and $\mathcal{Z}y = H \cdot \mathcal{Z}u$, where $H = \mathcal{Z}h$ is the transfer matrix.

Exercise 2[6] Consider once more the series and the parallel connections from Exercise 1, Tutorial 8 and Exercise 3, Tutorial 10. Show that the transfer matrix of the series connection is the product of the two transfer matrices, and the transfer matrix of the parallel connection is the sum of the two transfer matrices.

Exercise 3[2+1+2] Consider $\dot{x} = Ax + Bu$. The controllability operator at time $\varepsilon > 0$ is given by

$$L: \quad \mathcal{U} := \mathcal{L}^2([0,\varepsilon], \mathbb{R}^m) \to \mathbb{R}^n, \quad u(\cdot) \mapsto \int_0^\varepsilon e^{A(\varepsilon - t)} Bu(t)dt.$$

It maps an input function to the state at time ε when the system is started in $x(0) = 0$. The system is completely reachable (in time ε) if and only if L is surjective.

1. Show that

   $$L^*: \mathbb{R}^n \to \mathcal{U}, \quad x \mapsto \left\{ \begin{array}{ccc} [0,\varepsilon] & \to & \mathbb{R}^m \\ t & \mapsto & B^T e^{A^T(\varepsilon - t)} x \end{array} \right.$$

   is the adjoint of L.

2. Check that $LL^* = W(\varepsilon)$, the controllability Gramian (at time ε).

3. Recall that the system is completely reachable if and only if $W(\varepsilon)$ is invertible. Show that a special input that steers the system from 0 to x (in time ε) is given by

   $$\hat{u} = L^* W(\varepsilon)^{-1} x.$$

   This $\hat{u}$ has minimal "energy", that is, we have $\|u\|_{\mathcal{U}}^2 \geq \|\hat{u}\|_{\mathcal{U}}^2 = x^T W(\varepsilon)^{-1} x$ for all $u \in \mathcal{U}$ that steer the system from 0 to x in time ε. Equality holds if and only if $u = \hat{u}$.