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Control Theory – Tutorial 2

To be handed in till: Monday, May 04
To be discussed on: Tuesday, May 05, **15.45 - 17.15**

Exercise 1 (6 points)
A door-closing mechanism is supposed to automatically shut a door after it has been pulled open. In its simplest form, it consists of a mass-spring-damper system that can be modelled by

$$m\ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = u(t),$$

where m is the mass of the door (plus door-closing mechanism), c is the damping coefficient, k is the spring constant, $\theta(\cdot)$ is the door's opening angle, and $u(\cdot)$ corresponds to an external force (e.g., someone pushing the door). In this exercise, we study the autonomous case $u \equiv 0$, that is, the desired movement of the door after the external force is no longer acting. The designer of the mechanism can (to some extent) choose the parameters m, c, k, which are all real and positive, for physical reasons.
Let $\zeta := \frac{c}{2\sqrt{mk}}$ and $\omega := \sqrt{\frac{k}{m}}$. The "characteristic frequencies" of the system are the $\lambda \in \mathbb{C}$ such that $e^{\lambda t}$ solves the differential equation. Check that

$$\lambda_{1,2} = \omega(-\zeta \pm \sqrt{\zeta^2 - 1}).$$

By rescaling the time axis, we can reduce to the case where $\omega = 1$, without loss of generality. Discuss the qualitative behavior of $\theta(\cdot)$ for $\zeta > 1$, $\zeta = 1$, and $\zeta < 1$ (called the overdamped, critically damped, and underdamped case, respectively), e.g., by plotting the solutions for $\theta(0) = 1, \dot{\theta}(0) = 0$ for various values of ζ around 1. What would you call a "good" door closing mechanism?
Possible issues to be considered: how many times does the door swing considerably, how far does the door open to the opposite side (overshoot), is the convergence to zero monotone, how long does it take till the door can be considered closed?

Exercise 2 (2+4 points)

(a) Let R be a commutative ring, and let n, m be positive integers. Let $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$ and $D \in R^{m\times m}$ be given, and let A be invertible. Show that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \cdot \det(D - CA^{-1}B).$$

(b) Now let $R = R_0[t]$ for a commutative ring R_0 and let d, p be positive integers. Let $P_0, \ldots, P_d \in R^{p \times p}$ be given. Define the $dp \times dp$ matrices

$$K := \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & P_d \end{bmatrix} \quad \text{and} \quad L := \begin{bmatrix} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \\ -P_0 & -P_1 & \ldots & -P_{d-1} \end{bmatrix}$$

Show that $\det(K - tL) = \det(P_0 t^d + P_1 t^{d-1} + \ldots + P_{d-1} t + P_d) \in R_0[t]$.

Conclude that $\det(sK - L) = \det(P_0 + P_1 s + \ldots + P_d s^d) \in R_0[s]$ for $P_0, \ldots, P_d \in R_0^{p \times p}$.

Exercise 3 (2+2+4 points)

(a) Let D be a distribution and let a be a smooth function. Show that $(\dot{aD}) = \dot{a}D + a\dot{D}$.

(b) Let a be a smooth function. Show that $a\delta = a(0)\delta$, where δ is the Dirac delta distribution. Compute $a\dot{\delta}$.

(c) Compute the second derivative (in the distributional sense) of the functions given by

(i) $f(t) = |t|$

(ii) $f(t) = \sin(t)h(t)$, where h is the Heaviside function.

(iii) $f(t) = \cos(t)h(t)$, where h is the Heaviside function.

(iv) $f(t) = \lfloor t \rfloor$, where $\lfloor t \rfloor$ denotes the greatest integer less than or equal t.

Test your result using Maple, for example,
`simplify(diff(sin(t)*Heaviside(t),t$2));`