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Control Theory – Tutorial 3

To be handed in till: Monday, May 11
To be discussed on: Tuesday, May 12

Exercise 1 (1+1+2+2+1 points)
Consider $\dot{x}(t) = a(t)x(t) + b(t)$, where $a : \mathbb{R} \to \mathbb{R}$ is continuous, and $b : \mathbb{R} \to \mathbb{R}$ is locally integrable.

1. Show that if b is not continuous, there exists no classical solution $x \in \mathcal{C}^1(\mathbb{R})$.

2. Show that if x is locally integrable, then so is f, defined by $f(t) := a(t)x(t) + b(t)$.

3. The idea of a *weak solution* is motivated by considering a classical solution, multiplying both sides of the differential equation by a test function φ, integrating both sides from $-\infty$ to ∞, using partial integration and the compact support of φ. Show that the result of this procedure is

   $$\int_{-\infty}^{\infty} [x(t)\dot{\varphi}(t) + (a(t)x(t) + b(t))\varphi(t)]dt = 0.$$

   One says that $x \in L^1_{\text{loc}}(\mathbb{R})$ is a weak solution to $\dot{x} = ax + b$ if this is true for all test functions φ. (The advantage of the weak formulation is that $\dot{x}$ does not appear, that is, we don't need to restrict to differentiable x.)

   Remark: Every classical solution is a weak solution, but not conversely. A weak solution corresponds to a regular distributional solution: $\dot{D}_x = D_{ax+b}$. If a is smooth, we can even write $\dot{D}_x = aD_x + D_b$. In fact, the idea of weak solutions is the motivation for the concept of regular distributions.

4. Solve $\dot{x} = ax + h$, where $0 \neq a \in \mathbb{R}$ and h is the Heaviside function, by computing classical solutions on $(-\infty, 0)$ and on $(0, \infty)$ and by combining them into a global solution $x \in \mathcal{C}^0(\mathbb{R})$. Test your result using the Maple command
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   dsolve(diff(x(t),t)=a*x(t)+Heaviside(t),x(t));
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5. Show that the result of (d) is a weak solution to $\dot{x} = ax + h$.

Exercise 2 (6 points)
Consider a Jordan block

$$A = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix},$$

where $\lambda \in \mathbb{C}$. Show that if $\lambda \neq 0$, then

$$(A^t)_{jk} = \binom{t}{k-j} \lambda^{t+j-k}$$

for all $t \in \mathbb{N}$, where the binomial coefficients are defined such that $\binom{t}{s} := 0$ for $s < 0$ and $s > t$. If $\lambda = 0$, then

$$(A^t)_{jk} = \begin{cases} 1 & \text{if } t = k - j \\ 0 & \text{otherwise} \end{cases}$$

for all $t \in \mathbb{N}$. Conclude that

$$(e^{At})_{jk} = \begin{cases} \frac{t^{k-j}}{(k-j)!} e^{\lambda t} & \text{if } j \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 3 (5+2 points)
Let $K, L \in \mathbb{R}^{n \times n}$ be such that $\det(sK - L) \in \mathbb{R}[s] \backslash \{0\}$. Let

$$UKV = \begin{bmatrix} \mathrm{I}_\nu & 0 \\ & N \end{bmatrix}, \quad ULV = \begin{bmatrix} A & 0 \\ & \mathrm{I}_{n-\nu} \end{bmatrix}$$

be a Kronecker-Weierstraß form.

1. Show that the following data are uniquely determined by K, L:
   (a) ν, the size of A
   (b) $\mathrm{Spec}(A) = \{\lambda \in \mathbb{C} \mid \det(\lambda \mathrm{I} - A) = 0\}$
   (c) $\nu(\lambda) = \dim(\ker(\lambda \mathrm{I} - A))$, the geometric multiplicity of $\lambda \in \mathrm{Spec}(A)$
   (d) $\kappa = \min\{k \in \mathbb{N} \mid N^k = 0\}$, the nilpotent index of N.
2. Use Maple to compute a Kronecker-Weierstraß form of

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$