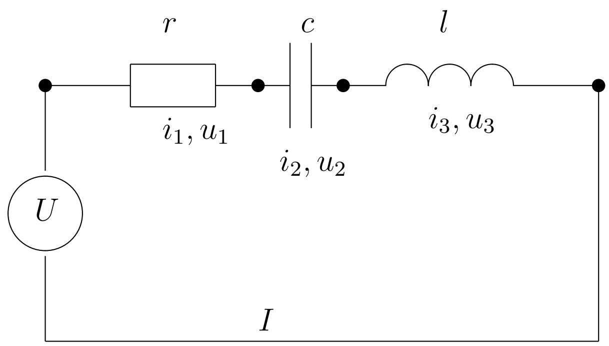
Eva Zerz
Kristina Schindelar SS 2009

Control Theory – Tutorial 4

To be handed in till: Monday, May 18
To be discussed on: Tuesday, May 19

Exercise 1 (2+2) Consider the following electrical circuit:



Here, U, I are the overall voltage/current of the network, whereas u_j, i_j for $j = 1, 2, 3$ represent the voltages/currents along the various components of the circuit as indicated in the sketch. There are 3 physical parameters which are real and positive constants (r is a resistance, c is a capacitance, l is an inductance). Kirchhoff's laws imply that

$$u_1 + u_2 + u_3 + U = 0 \quad \text{and} \quad i_1 = i_2 = i_3 = I.$$

Ohm's law says

$$u_1 = r i_1$$

and finally, we have the differential equations

$$c\frac{du_2}{dt} = i_2 \quad \text{and} \quad l\frac{di_3}{dt} = u_3.$$

Determine the differential equation that describes the relation between U and I alone. Do this

1. by hand, i.e., by manipulating the equations such that all u_j, i_j disappear;
2. systematically, using the elimination of latent variables and maybe a computer algebra system.

Exercise 2 (2+2+2+2+2) Let $\lambda \in \mathbb{C}$.

1. Show that for all $m, k \in \mathbb{N}$
$$\tfrac{d^m}{dt^m} t^k e^{\lambda t} = e^{\lambda t} (\tfrac{d}{dt} + \lambda)^m t^k.$$

2. Conclude that for all $P \in \mathbb{C}[s]$, $a \in \mathbb{C}[t]$
$$P(\tfrac{d}{dt}) a(t) e^{\lambda t} = e^{\lambda t} P(\tfrac{d}{dt} + \lambda) a(t).$$

3. Consider $P = (s - \lambda)^m \in \mathbb{C}[s]$, and $a \in \mathbb{C}[t]$. Show that $y(t) = a(t)e^{\lambda t}$ is a solution to $P(\frac{d}{dt})y = 0$ if and only if $\deg(a) \leq m - 1$.

   Remark: If $\lambda \in \mathbb{R}$ and y is required to be real-valued, a has to have real coefficients. The exercise shows that a solution is uniquely determined by choosing the $m = \deg(P)$ real coefficients of a.

4. Suppose that $\lambda_1 \neq \lambda_2$. Show that $(\frac{d}{dt} - \lambda_1)^{m_1}(\frac{d}{dt} - \lambda_2)^{m_2} y = 0$ if and only if y can be written in the form $y = y_1 + y_2$, where $(\frac{d}{dt} - \lambda_1)^{m_1} y_1 = 0$ and $(\frac{d}{dt} - \lambda_2)^{m_2} y_2 = 0$.

5. Conclude that for $\lambda \notin \mathbb{R}$ and $P = (s - \lambda)^m (s - \overline{\lambda})^m$, the function $y(t) = a(t)e^{\lambda t} + b(t)e^{\overline{\lambda} t}$ is a real-valued solution to $P(\frac{d}{dt})y = 0$ if and only if $\deg(a) \leq m - 1$ and $b = \overline{a}$.

   Remark: This shows that for $\lambda \notin \mathbb{R}$, a real-valued solution is uniquely determined by choosing the m complex coefficients of a, corresponding to a real vector space dimension of $2m = \deg(P)$.

Exercise 3 (2+2+2) Consider the system given by

$$\begin{aligned} \dddot{y}_1 + \dot{y}_1 + \ddot{y}_2 + \dot{y}_2 + y_2 &= 0 \\ \dot{y}_1 + 2y_1 + y_2 &= 0. \end{aligned}$$

Compute the dimension of its solution space. Confirm the result by manipulating the system equations (hint: add suitable consequences of the second equation to the first equation such that the highest derivative disappears). Compute a first order latent variable description of this system

1. in the "naive" way, i.e., setting $\xi = [y_1, y_2, \dot{y}_1, \dot{y}_2, \ddot{y}_1, \ddot{y}_2]^T$, which gives a representation of size 6;

2. by applying Theorem 2.22 to each of the two equations and by combining the results (this should give a representation of size 4).

3. Using the result of the manipulation above, find an even smaller representation.