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Control Theory – Tutorial 5

To be handed in till: Monday, May 25, To be discussed on: Tuesday, May 26

Exercise 1 [1+2+2+1+3] Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. For $A \in \mathbb{K}^{k\times l}$ and $B \in \mathbb{K}^{m\times n}$, one defines the *Kronecker (or: tensor) product*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1l}B \\ \vdots & & \vdots \\ a_{k1}B & \dots & a_{kl}B \end{bmatrix} \in \mathbb{K}^{km\times ln}.$$

Moreover, $\mathrm{vec}(A)$ is the column vector obtained by stacking the columns of A above each other,

$$\mathrm{vec}(A) = [a_{11}, \dots, a_{k1}, a_{12}, \dots, a_{k2}, \dots, a_{1l}, \dots, a_{kl}]^T \in \mathbb{K}^{kl}.$$

(a) Show that for any matrices X, Y such that AX and BY are defined,

$$(A \otimes B)(X \otimes Y) = AX \otimes BY.$$

(b) Conclude that if A, B are square, then

$$\mathrm{spec}(A \otimes B) = \{\lambda\mu \mid \lambda \in \mathrm{spec}(A), \mu \in \mathrm{spec}(B)\}.$$

Hint: $A \otimes B$ is similar to the Kronecker product of the Jordan forms of A, B.

(c) Show that for any three matrices A, X, B such that the matrix product AXB is defined, we have

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X).$$

(d) Show that the *Stein equation*

$$AXB - X = C$$

(where $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{m\times m}$, $C \in \mathbb{K}^{n\times m}$ are given) has a unique solution $X \in \mathbb{K}^{n\times m}$ if and only if $\lambda\mu \neq 1$ for all $\lambda \in \mathrm{spec}(A), \mu \in \mathrm{spec}(B)$.

Remark: In the lecture, we have studied a special case of the Stein equation, namely $A^TPA - P = -Q$. If A is discrete-time asymptotically stable, we have $|\lambda| < 1$ for all $\lambda \in \mathrm{spec}(A) = \mathrm{spec}(A^T)$, and thus the product of two eigenvalues of A can never be equal to one.

(e) Similarly, check that the *Sylvester equation*

$$AX + XB = C$$

(where A, B, C are given as above) has a unique solution $X \in \mathbb{K}^{n\times m}$ if and only if $\lambda + \mu \neq 0$ for all $\lambda \in \mathrm{spec}(A)$, $\mu \in \mathrm{spec}(B)$.

Remark: In the lecture, we have studied the special Sylvester equation $A^T P + PA = -Q$. If A is continuous-time asymptotically stable, we have $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \mathrm{spec}(A) = \mathrm{spec}(A^T)$, and thus the sum of two eigenvalues of A can never be equal to zero.

Exercise 2 [1+2+3] The use of Maple is recommended for the following computations.

1. Consider the system given in Exercise 3, Tutorial 4. Is it asymptotically stable?

2. Compute e^{At} for
$$A = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}.$$
Is the system given by $\dot{x} = Ax$ (asymptotically) stable? Compute x with $x_1(0) = 1$, $x_2(0) = 3$.

3. Show that $\dot{x} = Ax + Bu$, $y = Cx$, where
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \end{bmatrix},$$
is not an asymptotically stable state space system, but the associated input-output system (after elimination of the state x) is asymptotically stable. Any ideas on the reason for this?

Exercise 3 [5] Consider $\dot{x} = Ax$, $x(0) = x_0$, where $A \in \mathbb{R}^{n\times n}$ and $x_0 \in \mathbb{R}^n$. For a numerical solution, one studies "discretizations" of the differential equation. These are difference equations whose solutions are supposed to approximate the exact (continuous) solution. The simplest way to do this is by considering

$$\tfrac{\tilde{x}(t+h)-\tilde{x}(t)}{h} = A\tilde{x}(t)$$

for some "small" $h > 0$ (this is called Euler's method). Introducing the discrete time steps $t = ih$ for $i \in \mathbb{N}$, and setting $\hat{x}(i) := \tilde{x}(ih)$, we obtain

$$\hat{x}(i+1) = (I + hA)\hat{x}(i) \quad \text{for } i \in \mathbb{N}.$$

D.-t. asymptotic stability of this system is desirable, because it guarantees that the effect of a small error in x_0 will "die out" as i gets large.

Show that c.-t. asymptotic stability of A is necessary, but not sufficient for d.-t. asymptotic stability of $I + hA$. In fact, if A is c.-t. asymptotically stable, then $I + hA$ is d.-t. asymptotically stable if and only if

$$h < \min_{\lambda\in\mathrm{spec}(A)} \tfrac{2|\mathrm{Re}(\lambda)|}{|\lambda|^2}.$$