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Control Theory – Tutorial 6

To be handed in till: Monday, Jun. 08
To be discussed on: Tuesday, Jun. 09

Exercise 1[1+1+2+2+2] Show that the minimal "energy" needed to steer a controllable system $\dot{x} = Ax + Bu$ from 0 to $\bar{x}$ in time $\varepsilon > 0$ is given by

$$\min\{E(u) \mid \varphi(\varepsilon, 0, 0, u(\cdot)) = \bar{x}\} = \bar{x}^T W(\varepsilon)^{-1}\bar{x},$$

where $E(u) = \int_0^\varepsilon \|u(t)\|^2 dt$ and $W(\cdot)$ denotes the controllability Gramian

$$W(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau.$$

Proceed along the following steps:

1. Show that $\dot{W}(t) = AW(t) + W(t)A^T + BB^T$.
2. Show that $\frac{d}{dt}(X(t)^{-1}) = -X(t)^{-1}\dot{X}(t)X(t)^{-1}$ for every invertible matrix (simply differentiate $XX^{-1} = I$ using the product rule).
3. Consider $V(t) := x(t)^T W(t)^{-1} x(t)$, where $x(\cdot)$ is a solution of $\dot{x} = Ax + Bu$, and compute $\dot{V}(t)$.
4. Rewrite the result of step 3 as (omitting the argument t)
$$\dot{V} = -\|B^T W^{-1} x\|^2 + 2\langle u, B^T W^{-1} x\rangle.$$
Now use quadratic completion, integrate from 0 to ε, and conclude that for all u with $\varphi(\varepsilon, 0, 0, u(\cdot)) = \bar{x}$, we must have
$$\bar{x}^T W(\varepsilon)^{-1}\bar{x} \leq E(u).$$
5. Finally, show that equality is achieved for
$$u(t) = B^T e^{A^T(\varepsilon - t)} W(\varepsilon)^{-1}\bar{x}.$$

Remark: This explains the trade-off between the speed and the energy consumption of control: the smaller ε is, the larger is $\bar{x}^T W(\varepsilon)^{-1}\bar{x}$.

Exercise 2[3+4] The following physical explanations are just for those who are interested. One may just as well ignore them and go directly to the mathematical model.

1. An electric water kettle: Let v be the voltage applied to the kettle, and let x_1, x_2 be the temperatures of the heater coil and of the water, respectively. The change of x_1 is proportional to the electrical power fed into the system minus the coil's heat loss to the water; the electrical power is proportional to v^2; the heat loss is proportional to the temperature difference $x_1 - x_2$. The change of x_2 is proportional to the heat loss of the coil. This leads to the model

$$\begin{aligned} \dot{x}_1(t) &= av(t)^2 - b(x_1(t) - x_2(t)) \\ \dot{x}_2(t) &= c(x_1(t) - x_2(t)), \end{aligned}$$

where $a, b, c \in \mathbb{R}$. Setting $u := v^2$, this is a state space system. Discuss its stability and controllability in terms of a, b, c.

2. The "bipendulum": Consider a horizontal rod with a pendulum attached to each end. If the rod is moved horizontally, the two pendula will begin to swing. After some simplifications (pendulum = point mass, no friction, linearization for small oscillations of the pendula), the dynamics of this mechanical system is described by

$$\begin{aligned} \ddot{y}_1 + \omega_1^2 y_1 &= u \\ \ddot{y}_2 + \omega_2^2 y_2 &= u, \end{aligned}$$

where u is proportional to the acceleration of the rod (and can be seen as the input), y_i is the angle between the i-th pendulum and the vertical axis ($y = [y_1, y_2]^T$ takes the role of the output), and $\omega_i = \sqrt{g/L_i} > 0$ (g ... gravity constant, L_i ... length of i-th pendulum).

Transform the system into a state space system of size 4 (setting $x = [y_1, y_2, \dot{y}_1, \dot{y}_2]^T$ will do) and investigate the stability and controllability of this system. (Can you give a physical interpretation?)

Exercise 3[2+1+2]

1. With the notation introduced in the lecture, show that

$$\mathcal{R}(t, x) = \varphi(t, 0, x, 0) + \mathcal{R}(t)$$

and

$$\mathcal{C}(t, x) = \Phi(t)^{-1}(x + \mathcal{R}(t)),$$

where $\Phi(t)x = \varphi(t, 0, x, 0)$.

2. Conclude that in discrete time, $\mathcal{C} = (A^n)^{-1}\mathcal{R} \supseteq \mathcal{R}$. Thus, a discrete state space system is completely controllable to zero if and only if $\operatorname{im}(A^n) \subseteq \operatorname{im}(K)$, where K is the Kalman matrix.

   Remark: $(\cdot)^{-1}$ denotes the inverse image in (a) and (b).

3. Consider $P(\frac{d}{dt})y = u$, where P is a scalar monic polynomial of degree n. Transform this into a state space system of size n and show that the system is controllable.