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Control Theory – Tutorial 7

To be handed in till: Monday, Jun. 15
To be discussed on: Tuesday, Jun. 16

Exercise 1[3+3+2] Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ be such that (A, b) is a controllable matrix pair. Let the characteristic polynomial of A be

$$\chi_A = \det(sI_n - A) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0.$$

Recursively, define the following vectors: $v^{(n)} := b$ and

$$v^{(i)} := Av^{(i+1)} + a_i b \quad \text{for } i = n-1, \ldots, 0.$$

1. Show that $v^{(1)}, \ldots, v^{(n)}$ is a basis of $\mathbb{R}^n$ and $v^{(0)} = 0$.
2. Prove that the matrix $T := [v^{(1)}, \ldots, v^{(n)}]$ that has the vectors $v^{(i)}$ as columns, is a transformation matrix that puts (A, b) into controller form.
3. Apply this to

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Exercise 2[3+3]

1. Compute a Kalman controllability decomposition of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

   What are the uncontrollable modes of (A, b)? Is the system asymptotically controllable? (Consider both continuous and discrete time.)

2. Test the "bipendulum" system given by

$$\begin{aligned} \ddot{y}_1 + \omega_1^2 y_1 &= u \\ \ddot{y}_2 + \omega_2^2 y_2 &= u \end{aligned}$$

   for controllability directly (i.e., without reducing to state space form first).

Exercise 3[5] Prove that for a discrete system $x(t+1) = Ax(t) + Bu(t)$ with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the following are equivalent:

1. the system is completely controllable to zero;
2. $\text{im}(A^n) \subseteq \text{im}(K)$, where K is the Kalman matrix of (A, B);
3. if λ is an uncontrollable mode of (A, B), then $\lambda = 0$.

Exercise 4[1] Let

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$

be a state space realization of the system

$$\mathcal{B} = \{\begin{bmatrix} u \\ y \end{bmatrix} \mid \exists x : (1), (2) \text{ are satisfied}\}.$$

Show that

$$\mathcal{B} = \{\begin{bmatrix} u \\ y \end{bmatrix} \mid U_4(\frac{d}{dt})y = (-U_3B + U_4D)(\frac{d}{dt})u\},$$

where $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ is such that

$$U \begin{bmatrix} sI - A \\ C \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}.$$

Hint: Smith form, fundamental principle.