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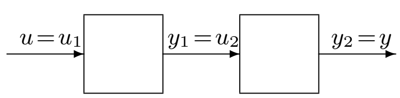
Control Theory – Tutorial 8

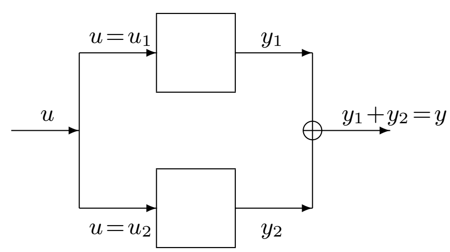
To be handed in till: Monday, Jun. 22
To be discussed on: Tuesday, Jun. 23

Exercise 1[2+2+3] Consider two state space systems

$$\begin{aligned}\dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i + D_i u_i,\end{aligned}$$

where $i = 1, 2$. Define the *series connection* of the two systems as the system arising from taking the output of the first system as the input of the second system (assuming $p_1 = m_2$). The *parallel connection* is defined by giving the same input to the two systems and summing their outputs (assuming $m_1 = m_2$ and $p_1 = p_2$).





Putting $x := [x_1^T, x_2^T]^T$, set up state space systems for both interconnections. Show the following:

(a) If the series connection is controllable, then both systems are controllable.

(b) Controllability of both systems is not sufficient for controllability of the series connection. However, it becomes sufficient under the additional assumption that for all $\lambda \in \operatorname{spec}(A_2)$, the matrix

$$\begin{bmatrix} A_1 - \lambda I & B_1 \\ C_1 & D_1 \end{bmatrix}$$

has full row rank.

(c) If the parallel connection is controllable, then both systems are controllable. The converse holds as well if we assume additionally that the spectra of A_1, A_2 are disjoint.

Exercise 2[2+3+2] Robust stability: Consider $\dot{x} = Ax$, where $A \in \mathbb{R}^{n\times n}$ is asymptotically stable. In practice, the entries of A are only known with a certain degree of accuracy, and thus we consider $\dot{x} = (A + \Delta)x$, where $\Delta \in \mathbb{C}^{n\times n}$ represents a perturbation of A. How large can Δ be without destroying the system's stability? Define

$$\mathfrak{U} := \{M \in \mathbb{C}^{n\times n} \mid \operatorname{spec}(M) \cap \overline{\mathbb{C}}_+ \neq \emptyset\},$$

where $\overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) \geq 0\}$. Thus $\mathfrak{U}$ is the set of complex matrices that are not asymptotically stable. Set

$$r(A) := \operatorname{dist}(A, \mathfrak{U}) := \inf\{\|A - M\| \mid M \in \mathfrak{U}\},$$

where $\|\cdot\|$ denotes the matrix norm associated to the Euclidean norm. Since $r(A)$ is the infimum of all $\|\Delta\|$ such that $A + \Delta$ is not asymptotically stable, one calls $r(A)$ the *stability radius* of A. Show that

$$r(A) = \frac{1}{\max_{\lambda\in\overline{\mathbb{C}}_+} \|G(\lambda)\|}, \tag{1}$$

where $G := (sI - A)^{-1} \in \mathbb{R}(s)^{n\times n}$, along the following steps:

(a) If Δ is such that $A + \Delta \in \mathfrak{U}$, then there exists $\lambda_0 \in \overline{\mathbb{C}}_+$ such that

$$\det(I - G(\lambda_0)\Delta) = 0.$$

Thus $0 = \min_{\|z\|=1} \|(I - G(\lambda_0)\Delta)z\|$.

(b) For all $A, B \in \mathbb{C}^{n\times n}$, we have

$$\min_{\|z\|=1} \|(A - B)z\| \geq \min_{\|z\|=1} \|Az\| - \max_{\|z\|=1} \|Bz\|.$$

Conclude that $\|\Delta\| \geq \frac{1}{\max_{\lambda\in\overline{\mathbb{C}}_+} \|G(\lambda)\|}$ for all Δ as in (a). Thus we have shown the inequality $\geq$ of (1).

(c) Now let $\lambda_0 \in \overline{\mathbb{C}}_+$ be such that

$$\max_{\lambda\in\overline{\mathbb{C}}_+} \|G(\lambda)\| = \|G(\lambda_0)\|.$$

Let $\mu > 0$ be the largest eigenvalue of $G(\lambda_0)G(\lambda_0)^*$. Show that

$$\Delta_0 := \frac{G(\lambda_0)}{\mu}$$

is such that $A + \Delta_0 \in \mathfrak{U}$ and $\|\Delta_0\| = \frac{1}{\|G(\lambda_0)\|}$, thus showing that equality is achieved in (1).

Exercise 3[3+3] From Exercise 3 of Tutorial 7, we know that the following are equivalent for a discrete system $x(t+1) = Ax(t) + Bu(t)$ with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$:

(a) The system is completely controllable to zero.

(b) If λ is an uncontrollable mode of (A, B), then $\lambda = 0$.

Show that these conditions are also equivalent to

(c) There exists $F \in \mathbb{R}^{m\times n}$ such that $A + BF$ is nilpotent.

Prove (c) $\Rightarrow$ (a) directly, using the feedback law $u(t) = Fx(t)$ (give also an explicit formula for $u(t)$), and (b) $\Rightarrow$ (c) using the pole shifting theorem.

Remark: This shows that the feedback law $u(t) = Fx(t)$ guarantees that $x(t) = 0$ for all $t \geq n$, independently of x_0. This strong type of feedback stabilization is called *deadbeat control.*

Compute all deadbeat controllers $F \in \mathbb{R}^{1\times 3}$ for

$$A = \begin{bmatrix} 0 & 1 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where $a_3, \ldots, a_6 \in \mathbb{R}$ are arbitrary.