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SS 2009

Control Theory – Tutorial 9

To be handed in till: Monday, Jun. 29
To be discussed on: Tuesday, Jun. 30

Exercise 1[2+2+2] Let (A, b) be a controllable matrix pair with $b \in \mathbb{R}^n$ ("single input").

1. Show that there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ and a feedback matrix $F \in \mathbb{R}^{1\times n}$ such that

$$\tilde{A} = T^{-1}AT + T^{-1}bF = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \text{ and } \tilde{b} := T^{-1}b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

   Remark: This means that by a coordinate transform $x = Tz$ and a feedback law $u = Fz + v$, the system $\dot{x} = Ax + bu$ can be brought into the form $\dot{z} = \tilde{A}z + \tilde{b}v$. This is a special case of the so-called *Brunovsky form*.

2. Show that $z_{n-i}^{(i+1)} = v$ for $i = 0, \ldots, n-1$, that is, if $z(0) = 0$, then

$$z_{n-i}(t) = \int_0^t \int_0^{t_i} \cdots \int_0^{t_1} v(\tau)d\tau dt_1 \cdots dt_i \text{ for } i = 0, \ldots, n-1.$$

3. Show that for a system in Brunovsky form, the optimal energy control function that steers the system from $z(0) = 0$ to $z(t_f) = z_f$ has the form $v(t) = c_0 + c_1 t + \ldots + c_{n-1}t^{n-1}$ for some $c_i \in \mathbb{R}$.

   Remark: Thus, it is particularly simple to compute an input v that steers a system in Brunovsky form from $z(0) = 0$ to $z(t_f) = z_f$: It amounts to solving a linear system of equations for the coefficients c_i. We have used this in Exercise 1 of Tutorial 1 already (for $n = 2$). From this v, one can then recompute u which steers the original system from $x(0) = 0$ to $x(t_f) = x_f = Tz_f$.

Exercise 2[3+2] Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $b \in \operatorname{im}(B)$ be given. Let $k \geq 1$ be the smallest integer such that

$$A^k b \in \mathcal{R}_k := \operatorname{im}[B, AB, \ldots, A^{k-1}B].$$

In particular, there exist $\hat{b}_i \in \operatorname{im}(B)$ such that

$$A^k b = \hat{b}_k + A\hat{b}_{k-1} + \ldots + A^{k-1}\hat{b}_1.$$

Define

$$x_1 := b \quad \text{and} \quad x_{i+1} := Ax_i - \hat{b}_i \text{ for } 1 \le i \le k.$$

1. Show that $x_1, \ldots, x_k$ are linearly independent and $x_{k+1} = 0$.
2. Conclude that there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $(A + BF)^i b$ for $0 \le i \le k-1$ are linearly independent, and $(A + BF)^k b = 0$.

   Hint: Write $\hat{b}_i = Bu_i$ and choose F such that $Fx_i = -u_i$ for $1 \le i \le k$.

Exercise 3[3] An expression of the type $sK - L \in \mathbb{R}[s]^{k\times l}$, where $K, L \in \mathbb{R}^{k\times l}$, is called a *pencil* of matrices. Two pencils $sK - L$ and $s\tilde{K} - \tilde{L}$ are called *equivalent* if there exist invertible matrices $U \in \mathbb{R}^{k\times k}, V \in \mathbb{R}^{l\times l}$ such that
$s\tilde{K} - \tilde{L} = U(sK - L)V$. Consider the matrix pencil

$$s[I, 0] - [A, B] = [sI - A, -B]$$

associated to $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. Show that two matrix pairs (A, B), $(\tilde{A}, \tilde{B})$ are feedback equivalent if and only if their associated pencils are equivalent.

Exercise 4[6]
Consider $\mathcal{S} = \{(A, B) \in \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 2} \mid (A, B) \text{ controllable, } \operatorname{rank}(B) = 2\}$. Show that all elements of $\mathcal{S}$ are feedback equivalent. Compute T, F, G that transform

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

into Brunovsky form.