Irreducible tensor products for alternating groups in characteristic 5

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Abstract

In this paper we study irreducible tensor products of representations of alternating groups and classify such products in characteristic 5.

1 Introduction

Let D_1 and D_2 be irreducible representations of a group G. In general the tensor product $D_1 \otimes D_2$ is not irreducible. We say that $D_1 \otimes D_2$ is a non-trivial irreducible tensor product if $D_1 \otimes D_2$ is irreducible and neither D_1 nor D_2 has dimension 1. The classification of non-trivial irreducible tensor products is relevant to the description of maximal subgroups in finite groups of Lie type, see [1] and [2].

Non-trivial irreducible tensor product of representations of symmetric groups have been fully classified (see [6], [13], [12], [28] and [30]). In particular non-trivial irreducible tensor products for S_n only exist if $p = 2$ and $n \equiv 2 \mod 4$. For alternating groups, non-trivial irreducible tensor products have been classified in characteristic 0 in [5] and in characteristic $p \geq 7$ in [7]. For covering groups of symmetric and alternating groups a partial classification of non-trivial irreducible tensor products can be found in [4], [8] and [24]. When considering groups of Lie type in defining characteristic, non-trivial irreducible tensor products are not unusual, due to Steinberg tensor product theorem. In non-defining characteristic however it has been proved that in almost all cases no non-trivial irreducible tensor products exist, see [25] and [26].

In this paper we will consider the case where $G = A_n$ is an alternating groups. Also we will mostly consider the case $p = 5$ in this paper, although some results hold in general, provided $p \neq 2$. Our main result is the following and extends the main theorem of [7]:

Theorem 1.1. *Let $p = 5$ and D_1 and D_2 be irreducible representations of A_n of dimension greater than 1. If $D_1 \otimes D_2$ is irreducible if and only if $n \not\equiv 0 \mod 5$ and, up to exchange, $D_1 \cong E^\lambda_\pm$ with $\lambda = \lambda^{\mathtt{M}}$ a JS-partition and $D_2 \cong E^{(n-1,1)}$ with $\mu \neq \mu^{\mathtt{M}}$. In this case $E^\lambda_\pm \otimes E^\mu \cong E^\nu$, where ν is obtained from λ by removing the top removable node and adding the bottom addable node.*

To prove the theorem we need to consider three cases:

(i) $D_1 = E^\lambda$ and $D_2 = E^\mu$: in this case $D_1 \otimes D_2$ is not irreducible by [6].

(ii) $D_1 = E^\lambda_\pm$ and $D_2 = E^\mu$: the proof of this case is covered by Theorems 7.3 and 7.4.

(iii) $D_1 = E^\lambda_\pm$ and $D_2 = E^\mu_\pm$: in this case $D_1 \otimes D_2$ is not irreducible by Theorem 8.3.

The first case in Theorem 1.1 appears also in larger characteristic (see [7] and Lemma 6.1). In smaller characteristic irreducible tensor products of the form $E^\lambda_\pm \otimes E^\mu_\pm$ exists. For example $E^{(3,2)}_+ \otimes E^{(3,2)}_- \cong E^{(4,1)}$ if $p = 2$ and $E^{(4,1^2)}_+ \otimes E^{(4,1^2)}_- \cong E^{(4,2)}$ if $p = 3$ (see [7]). For $p = 2$ and $p = 3$ partial classifications of irreducible tensor products can be found in [29].

2 Notations and basic results

Let F be an algebraically closed field of characteristic p.

For a partition $\lambda \vdash n$ let S^λ be the corresponding Specht module, $M^\lambda := \mathbf{1}{\uparrow}_{\Sigma_\alpha}^{\Sigma_n}$ to be the permutation module induced from the Young subgroup $\Sigma_\alpha = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \ldots \subseteq \Sigma_n$ and let Y^λ to be the corresponding Young module. (Notice that M^λ can be defined also for unordered partitions). If λ is a p-regular partition (that is a partition where no part is repeated p or more times) we define D^λ to be the irreducible $F\Sigma_n$-module indexed by λ. The modules D^λ, M^λ and Y^λ are known to be self-dual. Further, from their definition we have that $D^{(n)} \cong S^{(n)} \cong M^{(n)} \cong \mathbf{1}_{\Sigma_n}$. For more informations on such modules see [14], [15] and Section 4.6 of [27].

We have the following results about permutation and Young modules. For $\lambda \vdash n$ let $A_\lambda = \Sigma_\lambda \cap A_n$.

Lemma 2.1. *If $\lambda \vdash n$ with $\lambda \neq (1^n)$, then $M^\lambda{\downarrow}_{A_n} \cong 1{\uparrow}_{A_\lambda}^{A_n}$.*

It follows from Mackey's theorem

Lemma 2.2. *There exist indecomposable $F\Sigma_n$-modules $\{Y^\lambda \mid \lambda \vdash n\}$ such that $M^\lambda \cong Y^\lambda \oplus \bigoplus_{\mu \rhd \lambda} (Y^\mu)^{\oplus m_{\mu,\lambda}}$ for some $m_{\mu,\lambda} \in \mathbb{Z}_{\geq 0}$. Moreover, Y^λ can be characterized as the unique direct summand of M^λ such that $S^\lambda \subseteq Y^\lambda$. Finally, we have $(Y^\lambda)^* \cong Y^\lambda$ for all $\lambda \vdash n$.*

For a proof see [15] and [27, §4.6].

For any partition λ let $h(\lambda)$ be the number of parts of λ. For λ p-regular let $\lambda^{\mathtt{M}}$ be the Mullineux dual of λ, that is the partition with $D^{\lambda^{\mathtt{M}}} \cong D^\lambda \otimes \mathrm{sgn}$, where sgn is the sign representation of S_n. It is well known that, for $p \neq 2$, if $\lambda \neq \lambda^{\mathtt{M}}$ then $D^\lambda{\downarrow}_{A_n} = E^\lambda$ is irreducible (and in this case $E^\lambda \cong E^{\lambda^{\mathtt{M}}}$), while if $\lambda = \lambda^{\mathtt{M}}$ then $D^\lambda{\downarrow}_{A_n} = E^\lambda_+ \oplus E^\lambda_-$ is the direct sum of two non-isomorphic irreducible representations of A_n. Further all irreducible representations of A_n are of one of these two forms (see for example [11]).

Let M be a $F\Sigma_n$-module corresponding to a unique block B with content $(b_0, \ldots, b_{p-1})$ (see [21]). For $0 \leq i \leq p-1$, define e_iM as the restriction of $M{\downarrow}_{\Sigma_{n-1}}$ to the block with content $(b_0, \ldots, b_{i-1}, b_i - 1, b_{i+1}, \ldots, b_{p-1})$. Similarly, for $0 \leq i \leq p-1$, define f_iM as the restriction of $M{\uparrow}^{\Sigma_{n+1}}$ to the block with content $(b_0, \ldots, b_{i-1}, b_i + 1, b_{i+1}, \ldots, b_{p-1})$. Extend then the definition of e_iM and f_iM to arbitrary $F\Sigma_n$-modules additively. The following result holds for example by Theorems 11.2.7 and 11.2.8 of [21].

Lemma 2.3. *For M a $F\Sigma_n$-module we have that*

$$M{\downarrow}_{\Sigma_{n-1}} \cong e_0M \oplus \ldots \oplus e_{p-1}M \qquad \textit{and} \qquad M{\uparrow}^{\Sigma_{n+1}} \cong f_0M \oplus \ldots \oplus f_{p-1}M.$$

For $r \geq 1$ let $e_i^{(r)} : F\Sigma_n\text{-mod} \to F\Sigma_{n-r}\text{-mod}$ and $f_i^{(r)} : F\Sigma_n\text{-mod} \to F\Sigma_{n+r}\text{-mod}$ denote the divided power functors (see Section 11.2 of [21] for the definitions). For $r = 0$ define $e_i^{(0)}D^\lambda$ and $f_i^{(0)}D^\lambda$ to be equal to D^λ. The modules $e_i^rD^\lambda$ and $e_i^{(r)}D^\lambda$ (and similarly $f_i^rD^\lambda$ and $f_i^{(r)}D^\lambda$) are closely connected as we can be seen in the next two lemmas. For a partition λ and $0 \leq i \leq 1$ let $\varepsilon_i(\lambda)$ be the number of normal nodes of λ of residue i and $\varphi_i(\lambda)$ be the number of conormal nodes of λ of residue i (see Section 11.1 of [21] or Section 2 of [7] for two different but equivalent definitions of normal and conormal nodes). Normal and conormal nodes of partitions will play a crucial role throughout the paper.

If $\varepsilon_i(\lambda) \geq 1$ denote by $\tilde{e}_i(\lambda)$ the partition obtained from λ by removing the bottom normal node of residue i. Similarly, if $\varphi_i(\lambda) \geq 1$ denote by $\tilde{f}_i(\lambda)$ the partition obtained from λ by adding the top conormal node of residue i.

Lemma 2.4. *Let $\lambda \vdash n$ be a p-regular partition. Also let $0 \leq i \leq p-1$ and $r \geq 0$. Then $e_i^rD^\lambda \cong (e_i^{(r)}D^\lambda)^{\oplus r!}$. Further $e_i^{(r)}D^\lambda \neq 0$ if and only if $\varepsilon_i(\lambda) \geq r$. In this case*

(i) $e_i^{(r)}D^\lambda$ is a self-dual indecomposable module with head and socle isomorphic to $D^{\tilde{e}_i(\lambda)}$,

(ii) $[e_i^{(r)}D^\lambda : D^{\tilde{e}_i(\lambda)}] = \binom{\varepsilon_i(\lambda)}{r} = \dim \mathrm{End}_{\Sigma_{n-1}}(e_i^{(r)}D^\lambda)$,

(iii) if D^ψ is a composition factor of $e_i^{(r)}D^\lambda$ then $\varepsilon_i(\psi) \leq \varepsilon_i(\lambda) - r$, with equality holding if and only if $\psi = \tilde{e}_i(\lambda)$.

Lemma 2.5. *Let $\lambda \vdash n$ be a p-regular partition. Also let $0 \leq i \leq p-1$ and $r \geq 0$. Then $f_i^r D^\lambda \cong (f_i^{(r)} D^\lambda)^{\oplus r!}$. Further $f_i^{(r)} D^\lambda \neq 0$ if and only if $\varphi_i(\lambda) \geq r$. In this case*

(i) $f_i^{(r)} D^\lambda$ is a self-dual indecomposable module with head and socle isomorphic to $D^{\tilde{f}_i(\lambda)}$,

(ii) $[f_i^{(r)} D^\lambda : D^{\tilde{f}_i(\lambda)}] = \binom{\varphi_i(\lambda)}{r} = \dim \mathrm{End}_{\Sigma_{n+1}}(f_i^{(r)} D^\lambda)$,

(iii) if D^ψ is a composition factor of $f_i^{(r)} D^\lambda$ then $\varphi_i(\psi) \leq \varphi_i(\lambda) - r$, with equality holding if and only if $\psi = \tilde{f}_i(\lambda)$.

For proofs see Theorems 11.2.10 and 11.2.11 of [21] (the case $r = 0$ holds trivially). In particular, for $r = 1$, we have that $e_i = e_i^{(1)}$ and $f_i = f_i^{(1)}$. In this case there are other compositions factors of $e_i D^\lambda$ and $f_i D^\lambda$ which are known (see Remark 11.2.9 of [21]).

Lemma 2.6. *Let λ be a p-regular partition. If A is a normal node of λ of residue i and $\lambda \setminus A$ is p-regular then $[e_i D^\lambda : D^{\lambda \setminus A}]$ is equal to the number of normal nodes of λ of residue i weakly above A.*

Similarly if B is a conormal node of λ of residue i and $\lambda \cup B$ is p-regular then $[f_i D^\lambda : D^{\lambda \cup B}]$ is equal to the number of conormal nodes of λ of residue i weakly below B.

The following properties of e_i and f_i are just a special cases of Lemma 8.2.2(ii) and Theorem 8.3.2(i) of [21].

Lemma 2.7. *If M is self dual then so are $e_i M$ and $f_i M$.*

Lemma 2.8. *The functors e_i and f_i are left and right adjoint of each others.*

The first part of the next lemma follows from Lemma 5.2.3 of [21]. The second part follows by the definition of $\tilde{e}_i^r$ and $\tilde{f}_i^r$ and from Lemmas 2.4(iii) and 2.5(iii).

Lemma 2.9. *For $r \geq 0$ and p-regular partitions λ, ν we have that $\tilde{e}_i^r(\lambda) = D^\nu$ if and only if $\tilde{f}_i^r(\nu) = \lambda$. Further in this case $\varepsilon_i(\nu) = \varepsilon_i(\lambda) - r$ and $\varphi_i(\nu) = \varphi_i(\lambda) + r$.*

When considering the number of normal and conormal nodes of a partition we have the following result (see Lemma 2.8 of [28], for p-regular partitions it also follows from Lemmas 2.3, 2.4, 2.5 and Corollary 4.2 of [20]):

Lemma 2.10. *Any partition has 1 more conormal node than it has normal nodes.*

Since the modules e_iD^λ (or the modules f_iD^λ) correspond to pairwise distinct blocks we have the following result by Lemmas 2.3, 2.4 and 2.5.

Lemma 2.11. *For a p-regular partition $\lambda \vdash n$ we have that*

$$\dim \operatorname{End}_{\Sigma_{n-1}}(D^\lambda{\downarrow}_{\Sigma_{n-1}}) = \varepsilon_0(\lambda) + \ldots + \varepsilon_{p-1}(\lambda)$$

and

$$\dim \operatorname{End}_{\Sigma_{n+1}}(D^\lambda{\uparrow}^{\Sigma_{n+1}}) = \varphi_0(\lambda) + \ldots + \varphi_{p-1}(\lambda).$$

A p-regular partition $\lambda \vdash n$ for which $D^\lambda{\downarrow}_{\Sigma_{n-1}}$ is irreducible is called a JS-partition. JS-partitions can be classified as follow (see Section 4 of [17] and Theorem D of [18])

Lemma 2.12. *Let $\lambda = (a_1^{b_1}, \ldots, a_h^{b_h})$ with $a_1 > a_2 > \ldots > a_h \geq 1$ and $1 \leq b_i \leq p-1$ for $1 \leq i \leq h$. Then λ is a JS-partition if and only if $a_i - a_{i+1} + b_i + b_{i+1} \equiv 0 \mod p$ for each $1 \leq i < h$.*

For arbitrary modules $M_1, \ldots, M_h$ we will write $M \sim M_1|\ldots|M_h$ if M has a filtration with factors $M_1, \ldots, M_h$ counted from the bottom. For irreducible modules $D_1, \ldots, D_h$ we will write $M = D_1|\ldots|D_h$ if M is a uniserial module with composition factors $D_1, \ldots, D_h$ counted from the bottom.

3 Module structure

In the first part of this section we will consider the structure of certain permutation modules M^α.

Lemma 3.1. *Let $1 \leq k < p$ and $2k \leq n$. Then*

$$M^{(n-k,k)} \sim S^{(n-k,k)}|M^{(n-k+1,k-1)}.$$

Proof. See Lemmas 3.1 and 3.2 of [10]. $\square$

Lemma 3.2. *Let $p = 5$ and $n \equiv 1 \mod 5$ with $n \geq 6$. Then*

$$\begin{aligned}
Y^{(n)} &= D^{(n)} = S^{(n)}, \\
Y^{(n-1,1)} &= D^{(n-1,1)} = S^{(n-1,1)}, \\
Y^{(n-2,2)} &= \overbrace{D^{(n)}|D^{(n-2,2)}}^{S^{(n-2,2)}} \,|\, \overbrace{D^{(n)}}^{S^{(n)}}, \\
Y^{(n-3,3)} &= D^{(n-3,3)} = S^{(n-3,3)}, \\
Y^{(n-2,1^2)} &= D^{(n-2,1^2)} = S^{(n-2,1^2)}, \\
Y^{(n-3,2,1)} &\sim \overbrace{D^{(n-2,2)}|D^{(n-3,2,1)}}^{S^{(n-3,2,1)}} \,|\, \overbrace{D^{(n)}|D^{(n-2,2)}}^{S^{(n-2,2)}}, \\
Y^{(n-3,1^3)} &= D^{(n-3,1^3)} = S^{(n-3,1^3)},
\end{aligned}$$

Further

$$\begin{aligned}
M^{(n)} &\cong Y^{(n)},\\
M^{(n-1,1)} &\cong Y^{(n-1,1)} \oplus Y^{(n)},\\
M^{(n-2,2)} &\cong Y^{(n-2,2)} \oplus Y^{(n-1,1)},\\
M^{(n-3,3)} &\cong Y^{(n-3,3)} \oplus Y^{(n-2,2)} \oplus Y^{(n-1,1)},\\
M^{(n-2,1^2)} &\cong Y^{(n-2,1^2)} \oplus Y^{(n-2,2)} \oplus (Y^{(n-1,1)})^2,\\
M^{(n-3,2,1)} &\cong Y^{(n-3,2,1)} \oplus Y^{(n-2,1^2)} \oplus Y^{(n-3,3)} \oplus Y^{(n-2,2)} \oplus (Y^{(n-1,1)})^2,\\
M^{(n-3,1^3)} &\cong Y^{(n-3,1^3)} \oplus (Y^{(n-3,2,1)})^2 \oplus (Y^{(n-2,1^2)})^3 \oplus Y^{(n-3,3)}\\
&\quad \oplus Y^{(n-2,2)} \oplus (Y^{(n-1,1)})^3.
\end{aligned}$$

Proof. Notice first that all the considered simple modules correspond to pairwise distinct blocks, apart for $D^{(n)}$, $D^{(n-2,2)}$ and $D^{(n-3,2,1)}$ all three of which correspond to a single block. From Theorem 24.15 of [14] and from [16] we have that $[S^{(n-2,2)} : D^{(n)}] = 1$, $[S^{(n-3,2,1)} : D^{(n)}] = 0$ and $[S^{(n-3,2,1)} : D^{(n-2,2)}] = 1$. It follows that the structure of the Specht modules is as given in the lemma. Further, since the Young modules are indecomposable and self-dual it is easy to see that the Young modules structure is also as given in the lemma, apart possibly for the structure of $Y^{(n-3,2,1)}$.

From block decomposition we have that $D^{(n-3,2)} \cong S^{(n-3,2)}$ is a direct summand of $M^{(n-3,2)}$. In particular $D^{(n-3,2)}{\uparrow}^{S_n}$ is a direct summand of $M^{(n-3,2,1)}$. Notice that since $n \equiv 1 \mod 5$,

$$(n-3,2) = \begin{array}{|cccc|l}\hline 0 & 1 & 2 & \quad 2 & 3\\ \hline 4 & 0 & \multicolumn{1}{|l}{1} & & \\ \cline{1-2} \multicolumn{2}{l}{\;3} & & & \end{array} .$$

So, from Lemmas 2.3 and 2.5, from Corollary 17.14 of [14] and from block decomposition we have that

$$\begin{aligned}
D^{(n-3,2)}{\uparrow}^{S_n} &\cong S^{(n-3,2)}{\uparrow}^{S_n}\\
&\sim S^{(n-3,2,1)}|S^{(n-3,3)}|S^{(n-2,2)}\\
&\sim (\overbrace{S^{(n-3,2,1)}|S^{(n-2,2)}}^{f_3D^{(n-3,2)}}) \oplus \overbrace{S^{(n-3,3)}}^{f_1D^{(n-3,2)}} .
\end{aligned}$$

Since $f_3D^{(n-3,2)}$ is indecomposible by Lemma 2.5, it follows that $f_3D^{(n-3,2)} \cong Y^{(n-3,2,1)}$ by Lemma 2.2.

The multiplicities of the Young modules as direct summands of the modules M^α follow by comparing multiplicities of composition factors and from 14.1 of [14]. $\square$

Lemma 3.3. *Let $p = 5$ and $n \equiv 4 \mod 5$ with $n \geq 9$. Then*

$$\begin{aligned} Y^{(n)} &= D^{(n)} = S^{(n)}, \\ Y^{(n-1,1)} &= D^{(n-1,1)} = S^{(n-1,1)}, \\ Y^{(n-2,2)} &= D^{(n-2,2)} = S^{(n-2,2)}, \\ Y^{(n-3,3)} &= \overbrace{D^{(n-2,2)}|D^{(n-3,3)}}^{S^{(n-3,3)}} \mid \overbrace{D^{(n-2,2)}}^{S^{(n-2,2)}}, \\ Y^{(n-2,1^2)} &= D^{(n-2,1^2)} = S^{(n-2,1^2)}, \\ Y^{(n-3,2,1)} &= D^{(n-3,2,1)} = S^{(n-3,2,1)}, \\ Y^{(n-3,1^3)} &= D^{(n-3,1^3)} = S^{(n-3,1^3)}. \end{aligned}$$

Further

$$\begin{aligned} M^{(n)} &\cong Y^{(n)}, \\ M^{(n-1,1)} &\cong Y^{(n-1,1)} \oplus Y^{(n)}, \\ M^{(n-2,2)} &\cong Y^{(n-2,2)} \oplus Y^{(n-1,1)} \oplus Y^{(n)}, \\ M^{(n-3,3)} &\cong Y^{(n-3,3)} \oplus Y^{(n-1,1)} \oplus Y^{(n)}, \\ M^{(n-2,1^2)} &\cong Y^{(n-2,1^2)} \oplus Y^{(n-2,2)} \oplus (Y^{(n-1,1)})^2 \oplus Y^{(n)}, \\ M^{(n-3,2,1)} &\cong Y^{(n-3,2,1)} \oplus Y^{(n-2,1^2)} \oplus Y^{(n-3,3)} \oplus Y^{(n-2,2)} \oplus (Y^{(n-1,1)})^2 \oplus Y^{(n)}, \\ M^{(n-3,1^3)} &\cong Y^{(n-3,1^3)} \oplus (Y^{(n-3,2,1)})^2 \oplus (Y^{(n-2,1^2)})^3 \oplus Y^{(n-3,3)} \oplus (Y^{(n-2,2)})^2 \\ &\quad \oplus (Y^{(n-1,1)})^3 \oplus Y^{(n)}. \end{aligned}$$

Proof. Notice first that all the considered simple modules correspond to pairwise distinct blocks, apart for $D^{(n-2,2)}$ and $D^{(n-3,3)}$ which correspond to the same block. From Theorem 24.15 of [14] we have that $[S^{(n-3,3)} : D^{(n-2,2)}] = 1$. It follows that the structure of the Specht modules is as given in the lemma. Further, since the Young modules are indecomposable and self-dual it is easy to see that the Young modules structure is also as given in the lemma. The multiplicities of the Young modules as direct summands of the modules M^α follow by comparing multiplicities of composition factors and from 14.1 of [14]. $\square$

Lemma 3.4. *Let $p=5$ and $n \equiv 0 \mod 5$ with $n \geq 10$. Then*

$$\begin{aligned}
Y^{(n)} &= D^{(n)} = S^{(n)},\\
Y^{(n-1,1)} &= \overbrace{D^{(n)}|D^{(n-1,1)}}^{S^{(n-1,1)}} \,|\, \overbrace{D^{(n)}}^{S^{(n)}},\\
Y^{(n-2,2)} &= D^{(n-2,2)} = S^{(n-2,2)},\\
Y^{(n-3,3)} &= D^{(n-3,3)} = S^{(n-3,3)},\\
Y^{(n-4,4)} &= \overbrace{D^{(n-2,2)}|D^{(n-4,4)}}^{S^{(n-2,2)}} \,|\, \overbrace{D^{(n-2,2)}}^{S^{(n-2,2)}},\\
Y^{(n-2,1^2)} &\sim \overbrace{D^{(n-1,1)}|D^{(n-2,1^2)}}^{S^{(n-2,1^2)}} \,|\, \overbrace{D^{(n)}|D^{(n-1,1)}}^{S^{(n-1,1)}},\\
Y^{(n-3,2,1)} &= D^{(n-3,2,1)} = S^{(n-3,2,1)},\\
Y^{(n-4,3,1)} &= \overbrace{D^{(n-3,2,1)}|D^{(n-4,3,1)}}^{S^{(n-4,3,1)}} \,|\, \overbrace{D^{(n-3,2,1)}}^{S^{(n-3,2,1)}},\\
Y^{(n-4,2^2)} &= D^{(n-4,2^2)} = S^{(n-4,2^2)}.
\end{aligned}$$

Further

$$\begin{aligned}
M^{(n)} &\cong Y^{(n)},\\
M^{(n-1,1)} &\cong Y^{(n-1,1)},\\
M^{(n-2,2)} &\cong Y^{(n-2,2)} \oplus Y^{(n-1,1)},\\
M^{(n-3,3)} &\cong Y^{(n-3,3)} \oplus Y^{(n-2,2)} \oplus Y^{(n-1,1)},\\
M^{(n-4,4)} &\cong Y^{(n-4,4)} \oplus Y^{(n-3,3)} \oplus Y^{(n-1,1)},\\
M^{(n-2,1^2)} &\cong Y^{(n-2,1^2)} \oplus Y^{(n-2,2)} \oplus Y^{(n-1,1)},\\
M^{(n-3,2,1)} &\cong Y^{(n-3,2,1)} \oplus Y^{(n-2,1^2)} \oplus Y^{(n-3,3)} \oplus \big(Y^{(n-2,2)}\big)^2 \oplus Y^{(n-1,1)},\\
M^{(n-4,3,1)} &\cong Y^{(n-4,3,1)} \oplus Y^{(n-2,1^2)} \oplus Y^{(n-4,4)} \oplus \big(Y^{(n-3,3)}\big)^2 \oplus Y^{(n-2,2)}\\
&\qquad \oplus Y^{(n-1,1)},\\
M^{(n-4,2^2)} &\cong Y^{(n-4,2^2)} \oplus Y^{(n-4,3,1)} \oplus Y^{(n-3,2,1)} \oplus Y^{(n-2,1^2)} \oplus Y^{(n-4,4)}\\
&\qquad \oplus \big(Y^{(n-3,3)}\big)^2 \oplus \big(Y^{(n-2,2)}\big)^2 \oplus Y^{(n-1,1)}.
\end{aligned}$$

Proof. We have the following subsets of pairwise distinct blocks: $\{D^{(n)}, D^{(n-1,1)}, D^{(n-1^2)}\}$, $\{D^{(n-2,2)}, D^{(n-4,4)}\}$, $\{D^{(n-3,3)}\}$, $\{D^{(n-3,2,1)}, D^{(n-4,3,1)}\}$ and $\{D^{(n-4,2^2)}\}$. The structure of the Specht modules then follows by Theorems 24.1 and 24.15 of [14] and by [16]. Further, since the Young modules are indecomposable and self-dual it is easy to see that the Young modules structure is also as given in the lemma. The multiplicities of the Young modules as direct summands of the modules M^α follow by comparing multiplicities of composition factors and from 14.1 of [14]. $\square$

We will now prove that in certain cases there exists $\psi : M^{(n-4,2^2)} \to \mathrm{End}_F(D^\lambda)$ which does not vanish on $S^{(n-4,2^2)}$.

Lemma 3.5. *Let $p \geq 3$, $n \geq 6$ and V be a FS_n-module. If*

$$\begin{aligned}x_{2^2} = {} & (2,5)(3,6) - (3,5)(2,6) - (1,5)(3,6) + (1,6)(3,5) - (2,5)(1,6) \\ & + (1,5)(2,6) - (2,4)(3,6) + (3,4)(2,6) + (1,4)(3,6) - (1,6)(3,4) \\ & + (2,4)(1,6) - (1,4)(2,6) - (2,5)(3,4) + (3,5)(2,4) + (1,5)(3,4) \\ & - (1,4)(3,5) + (2,5)(1,4) - (1,5)(2,4)\end{aligned}$$

and $x_{2^2}V \neq 0$ then there exists $\psi : M^{(n-4,2^2)} \to \mathrm{End}_F(V)$ which does not vanish on $S^{(n-4,2^2)}$.

Proof. Let $\{v_{\{x,y\},\{z,w\}\}}|x, y, z, w \in \{1, \ldots, n\}$ distinct$\}$ be the standard basis of $M^{(n-4,2^2)}$. Define $\psi : M^{(n-4,2^2)} \to \mathrm{End}_F(V)$ through

$$\psi(v_{\{x,y\},\{z,w\}})(a) = (x, y)(z, w)a$$

for each $a \in V$. Let e be the basis element of $S^{(n-4,2^2)}$ corresponding to the tableau

$$\begin{array}{lllll} 1 & 4 & 7 & \ldots & n \\ 2 & 5 \\ 3 & 6 \end{array}$$

(see [14, Section 8] for definition of e). Then $\psi(e)(a) = 2x_{2^2}a$, from which the lemma follows. $\square$

Lemma 3.6. *Let $p = 5$, $n \geq 6$ and $\lambda \vdash n$ be 5-regular with $h(\lambda), h(\lambda^{\mathtt{M}}) \geq 3$. Then there exists $\psi : M^{(n-4,2^2)} \to \mathrm{End}_F(D^\lambda)$ which does not vanish on $S^{(n-4,2^2)}$.*

Proof. From Theorem 2.8 of [10] we have that $D^{(4,1^2)}$ or $D^{(3,1^3)}$ is a composition factor of $D^\lambda\downarrow_{\Sigma_6}$. So it is enough to prove that $x_{2^2}D^{(4,1^2)}$ and $x_{2^2}D^{(3,1^3)}$ are non-zero, where x_{2^2} is as in Lemma 3.5. Notice that $D^{(4,1^2)} \cong S^{(4,1^2)}$ and $D^{(3,1^3)} \cong S^{(3,1^3)}$. Let $\{v_{a,b}\}$, $\{e_{a,b}\}$, $\{v_{a,b,c}\}$ and $\{e_{a,b,c}\}$ be the standard bases of $M^{(4,1^2)}$, $S^{(4,1^2)}$, $M^{(3,1^3)}$ and $S^{(3,1^3)}$ respectively. It can be checked that $x_{2^2}e_{2,4}$ has non-zero coefficient for $v_{2,5}$ and that $x_{2^2}e_{2,3,4}$ has non-zero coefficient on $v_{1,5,6}$ and so the lemma holds. $\square$

We will now consider certain submodules of $D^\lambda{\downarrow}_{S_{n-2,2}}$. The next lemma generalizes Lemma 1.2 of [7] and will used in studying such restrictions.

Lemma 3.7. *Let $M_1, \ldots, M_h, X, Y$ be FG modules. Assume that that $M_1 \oplus \ldots \oplus M_h \subseteq X \oplus Y$ and that M_i has simple socle for each $1 \leq i \leq h$. Then there exist I_X, I_Y disjoint with $I_X \cup I_Y = \{1, \ldots, h\}$ such that, up to isomorphism, $\sum_{i \in I_X} M_i \subseteq X$ and $\sum_{i \in I_Y} M_i \subseteq Y$.*

Proof. Let π_X and π_Y be the projections to X and Y respectively. Since $\pi_X + \pi_Y = \mathrm{id}$ and the modules M_i have simple socles, we can find disjoint sets I_X, I_Y with $I_X \cup I_Y = \{1, \ldots, h\}$ such that π_X and π_Y are injective on $\sum_{i \in I_X} \mathrm{soc}(M_i)$ and $\sum_{i \in I_Y} \mathrm{soc}(M_i)$ respectively. It follows that π_1 and π_2 are injective on $\sum_{i \in I_X} M_i$ and $\sum_{i \in I_Y} M_i$ respectively and so the lemma holds. □

Lemma 3.8. *Let $p \geq 3$ and $\lambda \vdash n$ be p-regular. For $0 \leq i < p$ we have that $e_i^{(2)}(D^\lambda) \otimes (D^{(2)} \oplus D^{(1^2)})$ is a direct summand of $D^\lambda{\downarrow}_{S_{n-2,2}}$.*

Proof. From Lemma 2.4 we can assume that $\varepsilon_i(\lambda) \geq 2$ (else $e_i^{(2)} D^\lambda = 0$).

By definition $e_i^2 D^\lambda$ is a block component of $D^\lambda{\downarrow}_{S_{n-2}}$. Comparing block decomposition of $D^\lambda{\downarrow}_{S_{n-2}}$ and $D^\lambda{\downarrow}_{S_{n-2,2}}$, there exist a module M which is a direct sum of $D^\lambda{\downarrow}_{S_{n-2,2}}$ with $M{\downarrow}_{S_{n-2}} \cong e_i^2 D^\lambda$. Notice that M is the sum of the block components of $D^\lambda{\downarrow}_{S_{n-2,2}}$ corresponding to the blocks of $D^{\tilde{e}_i^2(\lambda)} \otimes D^{(2)}$ and of $D^{\tilde{e}_i^2(\lambda)} \otimes D^{(1^2)}$. From Lemmas 2.4 and Lemma 1.1 of [7] we have that

$$\mathrm{soc}(M){\downarrow}_{S_{n-2}} \cong \mathrm{soc}(e_i^2 D^\lambda) \cong D^{\tilde{e}_i^2(\lambda)} \oplus D^{\tilde{e}_i^2(\lambda)}.$$

We will first show that $\mathrm{soc}(M) \cong D^{\tilde{e}_i^2(\lambda)} \otimes (D^{(2)} \oplus D^{(1^2)})$. By definition of M, in order to do this it is enough to prove that

$$[\mathrm{soc}(D^\lambda{\downarrow}_{S_{n-2,2}}) : D^{\tilde{e}_i^2(\lambda)} \otimes D^{(2)}] = 1.$$

From Lemma 2.5, by definition of $f_i^{(2)}$ and considering block decomposition we have that

$$\begin{aligned}\dim \mathrm{Hom}_{S_{n-2,2}}(D^{\tilde{e}_i^2(\lambda)} \otimes D^{(2)}, D^\lambda{\downarrow}_{S_{n-2,2}}) &= \dim \mathrm{Hom}_{S_n}((D^{\tilde{e}_i^2(\lambda)} \otimes D^{(2)}){\uparrow}^{S_n}, D^\lambda) \\ &= \dim \mathrm{Hom}_{S_n}(f_i^{(2)}(D^{\tilde{e}_i^2(\lambda)}), D^\lambda) \\ &= 1.\end{aligned}$$

So $\mathrm{soc}(M) \cong D^{\tilde{e}_i^2(\lambda)} \otimes (D^{(2)} \oplus D^{(1^2)})$. Since $D^{(2)}$ and $D^{(1^2)}$ correspond to distinct blocks of S_2 and since S_2 is semisimple (as $p \geq 3$), we have that

$M \cong (M_1 \otimes D^{(2)}) \oplus (M_2 \otimes D^{(1^2)})$ for some modules M_1, M_2 with socle $D^{\tilde{e}_i^2(\lambda)}$. In particular

$$M_1 \oplus M_2 \cong M{\downarrow}_{S_{n-2}} \cong e_i^{(2)} D^\lambda \oplus e_i^{(2)} D^\lambda.$$

From Lemma 2.4 we have that $e_i^{(2)} D^\lambda$ is indecomposable. From Lemma 3.7 it follows that M_1 and M_2 are isomorphically contained in $e_i^{(2)} D^\lambda$ and so, comparing dimensions, that $M_1, M_2 \cong e_i^{(2)} D^\lambda$. □

Lemma 3.9. *Let $p \geq 3$ and $\lambda \vdash n$ be p-regular. For each j with $\varepsilon_j(\lambda) > 0$ and for each $i \neq j$ there exists $b_{i,j} \in \{D^{(2)}, D^{(1^2)}\}$ such that*

$$\sum_{\substack{j:\varepsilon_j(\lambda)>0 \\ i \neq j}} e_i(D^{\tilde{e}_j(\lambda)}) \otimes b_{i,j}$$

is both a submodule and a quotient of $D^\lambda{\downarrow}_{S_{n-2,2}}$.

Proof. Since $D^\lambda{\downarrow}_{S_{n-2,2}}$, $e_i(D^{\tilde{e}_j(\lambda)})$, $D^{(2)}$ and $D^{(1^2)}$ are self-dual it is enough to show that there exist $b_{i,j}$ such that

$$\sum_{\substack{j:\varepsilon_j(\lambda)>0 \\ i \neq j}} e_i(D^{\tilde{e}_j(\lambda)}) \otimes b_{i,j} \subseteq D^\lambda{\downarrow}_{S_{n-2,2}}.$$

Since $p \geq 3$, there exist M_1, M_2 with $D^\lambda{\downarrow}_{S_{n-2,2}} \cong (M_1 \otimes D^{(2)}) \oplus (M_2 \otimes D^{(1^2)})$. From Lemmas 2.3 and 2.4

$$\sum_{\substack{j:\varepsilon_j(\lambda)>0 \\ i \neq j}} e_i(D^{\tilde{e}_j(\lambda)}) \subseteq \sum_{i \neq j} e_i e_j D^\lambda \subseteq D^\lambda{\downarrow}_{S_{n-2}} \cong M_1 \oplus M_2.$$

and the modules $e_i(D^{\tilde{e}_j(\lambda)})$ have simple socle, if they are non-zero. The lemma then follows by Lemma 3.7. □

4 Dimensions of homomorphism rings

In this section we study the size of certain homomorphism rings, which will allow us later in Sections 7 and 8 to prove that in almost all cases the tensor product of two irreducible representations of A_n is not irreducible.

Lemma 4.1. *For any FS_n-module V and any $\alpha \vdash n$ we have that*

$$\dim \mathrm{Hom}_{S_n}(M^\alpha, \mathrm{End}_F(V)) = \dim \mathrm{End}_{S_\alpha}(V{\downarrow}_{S_\alpha}).$$

Proof. This follows by Frobenius reciprocity, since $M^\alpha = 1{\uparrow}_{S_\alpha}^{S_n}$. □

Lemma 4.2. *Let $G = S_n$ or $G = A_n$ and let V and W be FG-modules. For $\alpha \vdash n$ let $m_{V^*,\alpha}$ and $m_{W,\alpha}$ be such that there exist $\varphi_1^\alpha, \ldots, \varphi_{m_{V^*,\alpha}}^\alpha \in \mathrm{Hom}_G(M^\alpha, V^*)$ with $\varphi_1^\alpha|_{S^\alpha}, \ldots, \varphi_{m_{V^*,\alpha}}^\alpha|_{S^\alpha}$ linearly independent and that similarly there exist $\psi_1^\alpha, \ldots, \psi_{m_{W,\alpha}}^\alpha \in \mathrm{Hom}_G(M^\alpha, W)$ with $\psi_1^\alpha|_{S^\alpha}, \ldots, \psi_{m_{V^*,\alpha}}^\alpha|_{S^\alpha}$ linearly independent. Then*

$$\dim \mathrm{Hom}_G(V, W) \geq \sum_{\alpha \in A} m_{V^*,\alpha} m_{W,\alpha},$$

where A is the set of all p-regular partitions of n if $G = S_n$ or A is the set of p-regular partitions $\alpha \vdash n$ with $\alpha > \alpha^{\mathtt{M}}$ if $G = A_n$.

The order on partitions appearing in the lemma is the lexicographic order.

Proof. Notice first that if $\alpha \in A$ then M^α has a unique composition factor isomorphic to D^α (which is the head of S^α) and all other composition factors are of the form D^β with $\beta > \alpha$ (Corollary 12.2 of [14]) if $G = S_n$. If $G = A_n$ and $\alpha \in A$ then M^α has a unique composition factor isomorphic to E^α (which is the head of S^α) since in this case $\alpha > \alpha^{\mathtt{M}}$ and all other composition factors of M^α are of the form E^β or $E^\beta_\pm$ for some $\beta > \alpha$.

Fix $\alpha \in A$ and let $0 \neq \varphi \in \langle \varphi_i^\alpha \rangle$ and $0 \neq \psi \in \langle \psi_j^\alpha \rangle$. Then φ and ψ do not vanish on S^α, in particular D^α or E^α is a composition factor of $\mathrm{Im}(\varphi)$ and of $\mathrm{Im}(\psi)$. It then follows that D^α or E^α is a composition factor of $\mathrm{Im}(\psi \circ \varphi^*)$ (in particular $\psi \circ \varphi^*$ is non-zero) and all other composition factors of $\mathrm{Im}(\psi \circ \varphi^*)$ are of the form D^β or E^β, $E^\beta_\pm$ with $\beta > \alpha$.

It then follows that the functions $\psi_j^\alpha \circ (\varphi_i^\alpha)^*$, with $\alpha \in A$, $1 \leq i \leq m_{V^*,\alpha}$, $1 \leq j \leq m_{W,\alpha}$ are linearly independent and so the lemma holds. $\square$

The following lemmas will be used to prove that in certain cases there exists $\varphi \in \mathrm{Hom}_{S_n}(M^\alpha, \mathrm{End}_F(D^\lambda))$ which does not vanish on S^α.

Lemma 4.3. *Let $p = 5$ and $n \equiv \pm 1 \mod 5$ with $n \geq 6$. If $\lambda \vdash n$ and*

$$\begin{aligned}\dim \mathrm{End}_{S_{n-3}}(D^\lambda{\downarrow}_{S_{n-3}}) > 2 \dim \mathrm{End}_{S_{n-3,2}}(D^\lambda{\downarrow}_{S_{n-3,2}}) + \dim \mathrm{End}_{S_{n-2}}(D^\lambda{\downarrow}_{S_{n-2}})\\ - \dim \mathrm{End}_{S_{n-3,3}}(D^\lambda{\downarrow}_{S_{n-3,3}}) - \dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}})\\ - \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}}) + 1,\end{aligned}$$

then there exists $\psi \in \mathrm{Hom}_{S_n}(M^{(n-3,1^3)}, \mathrm{End}_F(D^\lambda))$ which does not vanish on $S^{(n-3,1^3)}$.

Proof. It follows from Lemmas 3.2 and 3.3. $\square$

Lemma 4.4. *Let $p = 5$ and $n \equiv 0 \mod 5$ with $n \geq 10$. If $\lambda \vdash n$ and*

$$\begin{aligned}&\dim \mathrm{End}_{S_{n-4,3}}(D^\lambda{\downarrow}_{S_{n-4,3}})+\dim \mathrm{End}_{S_{n-3,2}}(D^\lambda{\downarrow}_{S_{n-3,2}})+\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}})\\ &\qquad -\dim \mathrm{End}_{S_{n-3,3}}(D^\lambda{\downarrow}_{S_{n-3,3}})-\dim \mathrm{End}_{S_{n-2}}(D^\lambda{\downarrow}_{S_{n-2}})\\ &\quad <\dim \mathrm{End}_{S_{n-4,2^2}}(D^\lambda{\downarrow}_{S_{n-4,2^2}})\end{aligned}$$

then there exists $\psi \in \mathrm{Hom}_{S_n}(M^{(n-4,2^2)}, \mathrm{End}_F(D^\lambda))$ which does not vanish on $S^{(n-4,2^2)}$.

Proof. It follows from Lemma 3.4. □

Lemma 4.5. *Let $p \geq 3$, $n \geq 4$ and $\lambda \vdash n$ with $\lambda \neq (n), (n)^{\mathtt{M}}$. Then*

$$\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) > \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}}).$$

Proof. See Theorem 3.3 of [22]. □

We will now prove that, in most cases, the inequality in the previous lemma can be improved.

Lemma 4.6. *Let α and β be partitions such that α is obtained from β by removing an j-node. If $i \neq j$ then all normal i-nodes of β are also normal in α and all conormal i-nodes of α are also conormal in β.*

Proof. As $i \neq j$ all removable i-nodes of β are also removable in α and all addable i-nodes of α are also addable in β. The lemma then follows from the definition of normal and conormal nodes. □

Lemma 4.7. *Let $p \geq 3$ and $\lambda \vdash n$ be p-regular. If $\varepsilon_j(\lambda) > 0$. Then*

$$\begin{aligned}\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) &\geq \sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda)-1) + \sum_{\substack{j:\varepsilon_j(\lambda)>0\\ i\neq j}} \varepsilon_i(\tilde{e}_j(\lambda))\\ &\geq \sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda)-2+|\{j : \varepsilon_j(\lambda)>0\}|).\end{aligned}$$

Proof. From Lemma 2.3 we have that

$$D^\lambda{\downarrow}_{S_{n-2}} = \sum_{i,j} e_j e_i(D^\lambda) = \sum_i e_i^2(D^\lambda) \oplus \sum_{i\neq j} e_i e_j(D^\lambda).$$

From block decomposition and from Lemmas 3.8 and 3.9 we have that, for certain $b_{i,j} \in \{D^{(2)}, D^{(1^2)}\}$

$$B := \sum_i (e_i^{(2)} \otimes (D^{(2)} \oplus D^{(1^2)})) \oplus \sum_{\substack{j:\varepsilon_j(\lambda)>0\\ i\neq j}} e_i(D^{\tilde{e}_j(\lambda)}) \otimes b_{i,j}$$

is both a submodule and a quotient of $D^\lambda{\downarrow}_{S_{n-2,2}}$. In particular, from Lemma 2.4,

$$\begin{aligned}\dim\mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) &\geq \dim\mathrm{End}_{S_{n-2,2}}(B)\\ &\geq \sum_i \dim\mathrm{End}_{S_{n-2,2}}(e_i^{(2)}\otimes(D^{(2)}\oplus D^{(1^2)}))\\ &\quad+\sum_{\substack{j:\varepsilon_j(\lambda)>0\\ i\neq j}}\dim\mathrm{End}_{S_{n-2,2}}(e_i(D^\lambda_j)\otimes b_{i,j})\\ &=\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda)-1)+\sum_{\substack{j:\varepsilon_j(\lambda)>0\\ i\neq j}}\varepsilon_i(\tilde{e}_j(\lambda)).\end{aligned}$$

From Lemma 4.6 we also have that if $\varepsilon_j(\lambda)>0$ then $\varepsilon_i(\tilde{e}_j(\lambda))\geq\varepsilon_i(\lambda)$. So

$$\begin{aligned}&\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda)-1)+\sum_{\substack{j:\varepsilon_j(\lambda)>0\\ i\neq j}}\varepsilon_i(\tilde{e}_j(\lambda))\\ &\quad\geq\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda)-1)+\sum_{\substack{j:\varepsilon_j(\lambda)>0\\ i\neq j}}\varepsilon_i(\lambda)\\ &\quad=\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda)-2)+\sum_{j:\varepsilon_j(\lambda)>0}\sum_i\varepsilon_i(\lambda)\\ &\quad=\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda)-2+|\{j:\varepsilon_j(\lambda)>0\}|).\end{aligned}$$

□

A proof of the next lemma could also be obtained using Theorems 4.2 and 4.7 of [19].

Lemma 4.8. *For any partition λ and for any residue i,*

$$\varepsilon_i(\lambda)=\varepsilon_{-i}(\lambda^{\mathtt{M}})\quad\textit{and}\quad\varphi_i(\lambda)=\varphi_{-i}(\lambda^{\mathtt{M}}).$$

If $\varepsilon_i(\lambda)>0$ then $\tilde{e}_i(\lambda)^{\mathtt{M}}=\tilde{e}_{-i}(\lambda^{\mathtt{M}})$, while if $\varphi_i(\lambda)>0$ then $\tilde{f}_i(\lambda^{\mathtt{M}})=\tilde{f}_{-i}(\lambda^{\mathtt{M}})$.

Proof. This follows from Lemma 2.4 and by comparing block decomposition of $D^\lambda{\downarrow}_{S_{n-1}}$ and of $D^{\lambda^{\mathtt{M}}}{\downarrow}_{S_{n-1}}\cong D^\lambda{\downarrow}_{S_{n-1}}\otimes\mathrm{sgn}$ (or of $D^\lambda{\uparrow}^{S_{n+1}}$ and of $D^{\lambda^{\mathtt{M}}}{\uparrow}^{S_{n+1}}\cong D^\lambda{\uparrow}^{S_{n+1}}\otimes\mathrm{sgn}$). □

Lemma 4.9. *Let $p\geq 3$ and λ be p-regular. If λ has at least 3 normal nodes then*

$$\dim\mathrm{End}_{S_{n-2,2}}(D^\mu{\downarrow}_{S_{n-2,2}})>\dim\mathrm{End}_{S_{n-1}}(D^\mu{\downarrow}_{S_{n-1}})+1.$$

If further $\lambda=\lambda^{\mathtt{M}}$ then

$$\dim\mathrm{End}_{S_{n-2,2}}(D^\mu{\downarrow}_{S_{n-2,2}})>\dim\mathrm{End}_{S_{n-1}}(D^\mu{\downarrow}_{S_{n-1}})+2.$$

Proof. From Lemmas 2.11 and 4.7 it is enough to prove that

$$\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda) - 3 + |\{j : \varepsilon_j(\lambda) > 0\}|) > 1$$

or

$$\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda) - 3 + |\{j : \varepsilon_j(\lambda) > 0\}|) > 2$$

when λ has at least 3 normal nodes (the last inequality only when $\lambda = \lambda^{\mathtt{M}}$).

Assume first that $|\{j : \varepsilon_j(\lambda) > 0\}| = 1$ and let k with $\varepsilon_k(\lambda) > 0$. Then $\varepsilon_i(\lambda) \geq 3$ and so

$$\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda) - 3 + |\{j : \varepsilon_j(\lambda) > 0\}|) = \varepsilon_k(\lambda)(\varepsilon_k(\lambda) - 2) \geq \varepsilon_k(\lambda) \geq 3.$$

Assume next that $|\{j : \varepsilon_j(\lambda) > 0\}| = 2$ and let $k \neq l$ with $\varepsilon_k(\lambda), \varepsilon_l(\lambda) > 0$. We can assume that $\varepsilon_k(\lambda) \geq 2$. Then

$$\begin{aligned}\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda) - 3 + |\{j : \varepsilon_j(\lambda) > 0\}|) &= \varepsilon_k(\lambda)(\varepsilon_k(\lambda) - 1) + \varepsilon_l(\lambda)(\varepsilon_l(\lambda) - 1)\\ &\geq \varepsilon_k(\lambda)\\ &\geq 2.\end{aligned}$$

Assume now that $\lambda = \lambda^{\mathtt{M}}$. Then from Lemma 4.8, we have that $k = -l$ and $\varepsilon_k(\lambda) = \varepsilon_l(\lambda) \geq 2$. In this case

$$\begin{aligned}\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda) - 3 + |\{j : \varepsilon_j(\lambda) > 0\}|) &= \varepsilon_k(\lambda)(\varepsilon_k(\lambda) - 1) + \varepsilon_l(\lambda)(\varepsilon_l(\lambda) - 1)\\ &\geq 2\varepsilon_k(\lambda)\\ &\geq 4.\end{aligned}$$

Assume last that $|\{j : \varepsilon_j(\lambda) > 0\}| \geq 3$ and let k, l, h pairwise different with $\varepsilon_k(\lambda), \varepsilon_l(\lambda), \varepsilon_h(\lambda) > 0$. Then

$$\sum_i \varepsilon_i(\lambda)(\varepsilon_i(\lambda) - 3 + |\{j : \varepsilon_j(\lambda) > 0\}|) \geq \varepsilon_k(\lambda)^2 + \varepsilon_l(\lambda)^2 + \varepsilon_h(\lambda)^2 \geq 3.$$

□

Lemma 4.10. *Let $p \geq 3$, $n \geq 4$ and $\lambda = \lambda^{\mathtt{M}} \vdash n$ be a partition with exactly 2 normal nodes. If there exist $i \neq j$ with $\varepsilon_i(\lambda), \varepsilon_j(\lambda) \neq 0$ then $\tilde{e}_i(\lambda)$ and $\tilde{e}_j(\lambda)$ are not JS-partitions.*

Proof. Assume instead that $\tilde{e}_i(\lambda)$ and $\tilde{e}_j(\lambda)$ are JS-partitions. Then, from Lemmas 2.3 and 2.4, $D^\lambda{\downarrow}_{S_{n-2}}$ has only two composition factors. Since $\lambda = \lambda^{\mathtt{M}}$ it follows that

$$D^\lambda{\downarrow}_{S_{n-2,2}} \cong (D^\nu \otimes D^{(2)}) \oplus (D^{\nu^{\mathtt{M}}} \otimes D^{(1^2)})$$

for a certain partition ν. Due to Lemma 2.11 this contradicts Lemma 4.5. The lemma then follows from Lemma 4.8. □

Lemma 4.11. *Let $p \geq 3$ and λ be a p-regular partition with 2 normal nodes. Assume that there exist $i \neq j$ with $\varepsilon_i(\lambda), \varepsilon_j(\lambda) = 1$. If*

$$\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) = \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}}) + 1$$

then, up to exchange, $\tilde{e}_i(\lambda)$ is a JS-partition and $\tilde{e}_j(\lambda)$ has at most 2 normal nodes. Also $\lambda \neq \lambda^{\mathtt{M}}$.

Proof. From Lemma 2.4 we have that $\varepsilon_i(\tilde{e}_i(\lambda)) = \varepsilon_i(\lambda) - 1 = 0$ and similarly $\varepsilon_j(\tilde{e}_j(\lambda)) = 0$. So from Lemmas 2.11 and 4.7 and by assumption

$$\begin{aligned}\sum_k \varepsilon_k(\tilde{e}_i(\lambda)) + \sum_k \varepsilon_k(\tilde{e}_j(\lambda)) &= \sum_{k\neq i} \varepsilon_k(\tilde{e}_i(\lambda)) + \sum_{k\neq j} \varepsilon_k(\tilde{e}_j(\lambda)) \\ &\leq \dim \mathrm{End}_{S_{n-2,2}}(D^\mu{\downarrow}_{S_{n-2,2}}) \\ &= \dim \mathrm{End}_{S_{n-1}}(D^\mu{\downarrow}_{S_{n-1}}) + 1 \\ &= 3.\end{aligned}$$

So $\tilde{e}_i(\lambda)$ and $\tilde{e}_j(\lambda)$ have in total at most 3 normal nodes, from which the first part of the lemma follows. The second part follows then from Lemma 4.10. □

Lemma 4.12. *Let $p \geq 3$ and λ be a p-regular partition with 2 normal nodes. Assume that there exists i with $\varepsilon_i(\lambda) = 2$ and let ν to be obtained from λ by removing the top removable node of λ. If*

$$\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) = \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}}) + 1$$

then $\tilde{e}_i(\lambda)$ is a JS-partition and ν is either a JS-partition or it is not p-regular. Also $\lambda \neq \lambda^{\mathtt{M}}$.

Proof. Notice first that from Lemma 3.8

$$D^\lambda{\downarrow}_{S_{n-2,2}} = (e_i^{(2)}(D^\lambda) \otimes (D^{(2)} \oplus D^{(1^2)})) \oplus M$$

for a certain module M. Comparing block decompositions of $D^\lambda{\downarrow}_{S_{n-2}}$ and $D^\lambda{\downarrow}_{S_{n-2,2}}$ we have that

$$M{\downarrow}_{S_{n-2}} \cong \sum_{(j,k)\neq(i,i)} e_j e_k(D^\lambda).$$

Also from Lemma 2.4

$$\dim \mathrm{End}_{S_{n-2,2}}(e_i^{(2)}(D^\lambda)\otimes(D^{(2)}\oplus D^{(1^2)})) = \varepsilon_i(\lambda)(\varepsilon_i(\lambda)-1) = 2.$$

Notice that M is self-dual, since it is the sum of certain block components of $D^\lambda{\downarrow}_{S_{n-2,2}}$. So, if M is non-zero and not simple, then $\dim \mathrm{End}_{S_{n-2,2}}(M) \geq 2$ (simple modules of $S_{n-2,2}$ are also self-dual) and so from Lemma 2.11

$$\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) \geq 2+2 > 3 = \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}})+1,$$

contradicting the assumptions. As all simple Σ_2-modules are 1-dimensional, M is non-zero and not simple if and only if $M{\downarrow}_{S_{n-2,2}} \cong \sum_{(j,k)\neq(i,i)} e_j e_k(D^\lambda)$ is non-zero and not simple. In order to prove the lemma it is then enough to prove that $\sum_{(j,k)\neq(i,i)} e_j e_k(D^\lambda)$ is non-zero and not simple, when λ is not as in the text of the lemma.

First assume that $\tilde{e}_i(\lambda)$ is not a JS-partition. Then, from Lemma 2.9, there exist $l \neq i$ with $\varepsilon_l(\lambda_i) \geq 1$. So, from Lemma 2.4,

$$\begin{aligned}
[\sum_{(j,k)\neq(i,i)} e_j e_k(D^\lambda) : D^{\tilde{e}_l\tilde{e}_i(\lambda)}] &\geq [e_i(D^\lambda) : D^{\tilde{e}_i(\lambda)}]\cdot[e_l(D^{\tilde{e}_i(\lambda)}) : D^{\tilde{e}_l\tilde{e}_i(\lambda)}]\\
&= \varepsilon_i(\lambda)\varepsilon_l(\tilde{e}_i(\lambda))\\
&\geq 2.
\end{aligned}$$

In particular $\sum_{(j,k)\neq(i,i)} e_j e_k(D^\lambda)$ is non-zero and not simple.

Assume next that ν is p-regular but not a JS-partition. From Lemmas 2.4 and 2.6 we have that D^ν is a composition component of $e_i(D^\lambda)$ and that $\varepsilon_i(\nu) \leq \varepsilon_i(\lambda)-2 = 0$. So $\sum_{l\neq i}\varepsilon_j(\nu) \geq 2$ and then

$$\begin{aligned}
\sum_{l\neq i}[\sum_{(j,k)\neq(i,i)} e_j e_k(D^\lambda) : D^{\tilde{e}_l(\nu)}] &\geq \sum_{l\neq i}[e_l(D^\lambda) : D^\nu]\cdot[e_l(D^\nu) : D^{\tilde{e}_l(\nu)}]\\
&\geq \sum_{l\neq i}\varepsilon_l(\nu)\\
&\geq 2.
\end{aligned}$$

So also in this case $\sum_{(j,k)\neq(i,i)} e_j e_k(D^\lambda)$ is non-zero and not simple.

Assume now that $\lambda = \lambda^{\mathtt{M}}$. Notice that $\nu = \lambda \setminus A$, where A is the top removable node of λ. Assume first that ν is not p-regular. Then $\lambda_1 = \lambda_p + 1$.This contradicts $\lambda = \lambda^{\mathtt{M}}$, by Lemma 2.2 of [3]. So we can assume that ν is p-regular. Further from Lemma 4.8 we have that $i = 0$, so that $\varepsilon_0(\nu) = 0$. In particular there exist $l \neq 0$ such that $e_l(D^\nu) \neq 0$. Since D^ν is a component of $e_0(D^\lambda)$, we then have that $e_l e_0(D^\lambda) \neq 0$. Since $\lambda = \lambda^{\mathtt{M}}$ we also have that $e_{-l}e_0(D^\lambda) \neq 0$. As $l \neq 0$ and so $l \neq -l$ as $p \geq 3$ is odd, it follows that $\sum_{(j,k)\neq(i,i)} e_j e_k(D^\lambda)$ is non-zero and not simple. □

Lemma 4.13. *Let $p \geq 3$. If $\lambda = \lambda^{\mathtt{M}}$ a p-regular partition and V is an FS_n-module, then*

$$\begin{aligned}&\dim \mathrm{Hom}_{A_n}(V{\downarrow_{A_n}}, \mathrm{Hom}_F(E^\lambda_+ \oplus E^\lambda_-, E^\lambda_\pm))\\ &\quad = \dim \mathrm{Hom}_{A_n}(\mathrm{Hom}_F(E^\lambda_\pm, E^\lambda_+ \oplus E^\lambda_-), V^*{\downarrow_{A_n}})\\ &\quad = \dim \mathrm{Hom}_{S_n}(V, \mathrm{End}_F(D^\lambda)).\end{aligned}$$

Proof. Using Frobenious reciprocity we have

$$\begin{aligned}\mathrm{Hom}_{A_n}(V{\downarrow_{A_n}}, \mathrm{Hom}_F(E^\lambda_+ \oplus E^\lambda_-, E^\lambda_\pm)) &\cong \mathrm{Hom}_{A_n}(V{\downarrow_{A_n}}, (E^\lambda_+ \oplus E^\lambda_-)^* \otimes E^\lambda_\pm)\\ &\cong \mathrm{Hom}_{S_n}(V, ((E^\lambda_+ \oplus E^\lambda_-)^* \otimes E^\lambda_\pm){\uparrow^{S_n}})\\ &\cong \mathrm{Hom}_{S_n}(V, (D^\lambda)^* \otimes D^\lambda)\\ &\cong \mathrm{Hom}_{S_n}(V, \mathrm{End}_F(D^\lambda)).\end{aligned}$$

The other equality holds similarly. □

5 Partitions of the from $(a + b, a)$ with b small

Partitions of the form $(a + b, a)$ with $0 \leq b \leq 3$ will play a special role in the proof of Theorem 1.1, since for these partitions Corollary 4.12 of [10] does not apply. So we will now study these partitions (and the corresponding simple modules and their restrictions to certain submodules of Σ_n) more in details.

Lemma 5.1. *Let $p = 5$ and $\lambda = (a + b, a) \vdash n$ with $0 \leq b \leq 3$. If $k \leq 4$ and*

$a \geq k$ then $D^\lambda{\downarrow}_{\Sigma_{n-k,k}}$ is given by

$$\begin{aligned}
D^{(a,a)}{\downarrow}_{\Sigma_{2a-1}} &\cong D^{(a,a-1)},\\
D^{(a,a)}{\downarrow}_{\Sigma_{2a-2,2}} &\cong (D^{(a,a-2)}\otimes D^{(2)})\oplus(D^{(a-1,a-1)}\otimes D^{(1^2)}),\\
D^{(a,a)}{\downarrow}_{\Sigma_{2a-3,3}} &\cong (D^{(a,a-3)}\otimes D^{(3)})\oplus(D^{(a-1,a-2)}\otimes D^{(2,1)}),\\
D^{(a,a)}{\downarrow}_{\Sigma_{2a-4,4}} &\cong (D^{(a-1,a-3)}\otimes D^{(3,1)})\oplus(D^{(a-2,a-2)}\otimes D^{(2^2)}),\\
D^{(a+1,a)}{\downarrow}_{\Sigma_{2a}} &\cong D^{(a+1,a-1)}\oplus D^{(a,a)},\\
D^{(a+1,a)}{\downarrow}_{\Sigma_{2a-1,2}} &\cong (D^{(a+1,a-2)}\otimes D^{(2)})\oplus(D^{(a,a-1)}\otimes D^{(2)})\oplus(D^{(a,a-1)}\otimes D^{(1^2)}),\\
D^{(a+1,a)}{\downarrow}_{\Sigma_{2a-2,3}} &\cong (D^{(a,a-2)}\otimes D^{(3)})\oplus(D^{(a,a-2)}\otimes D^{(2,1)})\oplus(D^{(a-1,a-1)}\otimes D^{(2,1)}),\\
D^{(a+1,a)}{\downarrow}_{\Sigma_{2a-3,4}} &\cong (D^{(a,a-3)}\otimes D^{(3,1)})\oplus(D^{(a-1,a-2)}\otimes D^{(3,1)})\oplus(D^{(a-1,a-2)}\otimes D^{(2^2)}),\\
D^{(a+2,a)}{\downarrow}_{\Sigma_{2a+1}} &\cong D^{(a+1,a)}\oplus D^{(a+2,a-1)},\\
D^{(a+2,a)}{\downarrow}_{\Sigma_{2a,2}} &\cong (D^{(a,a)}\otimes D^{(2)})\oplus(D^{(a+1,a-1)}\otimes D^{(2)})\oplus(D^{(a+1,a-1)}\otimes D^{(1^2)}),\\
D^{(a+2,a)}{\downarrow}_{\Sigma_{2a-1,3}} &\cong (D^{(a,a-1)}\otimes D^{(3)})\oplus(D^{(a,a-1)}\otimes D^{(2,1)})\oplus(D^{(a+1,a-2)}\otimes D^{(2,1)}),\\
D^{(a+2,a)}{\downarrow}_{\Sigma_{2a-2,4}} &\cong (D^{(a-1,a-1)}\otimes D^{(3,1)})\oplus(D^{(a,a-2)}\otimes D^{(3,1)})\oplus(D^{(a,a-2)}\otimes D^{(2^2)}),\\
D^{(a+3,a)}{\downarrow}_{\Sigma_{2a+2}} &\cong D^{(a+2,a)},\\
D^{(a+3,a)}{\downarrow}_{\Sigma_{2a+1,2}} &\cong (D^{(a+1,a)}\otimes D^{(2)})\oplus(D^{(a+2,a-1)}\otimes D^{(1^2)}),\\
D^{(a+3,a)}{\downarrow}_{\Sigma_{2a,3}} &\cong (D^{(a,a)}\otimes D^{(3)})\oplus(D^{(a+1,a-1)}\otimes D^{(2,1)}),\\
D^{(a+3,a)}{\downarrow}_{\Sigma_{2a-1,4}} &\cong (D^{(a,a-1)}\otimes D^{(3,1)})\oplus(D^{(a+1,a-2)}\otimes D^{(2^2)}).
\end{aligned}$$

Proof. For $k \leq 3$ see Lemmas 4.1, 4.5 and 4.7 of [10]. Further if $a \geq 4$, from the same lemmas,

$$\begin{aligned}
D^{(a,a)}\downarrow_{\Sigma_{2a-4,2^2}} \cong &(D^{(a-2,a-2)}\otimes D^{(2)}\otimes D^{(2)})\oplus(D^{(a-1,a-3)}\otimes D^{(2)}\otimes D^{(2)})\\
&\oplus(D^{(a-1,a-3)}\otimes D^{(1^2)}\otimes D^{(2)})\oplus(D^{(a-1,a-3)}\otimes D^{(2)}\otimes D^{(1^2)})\\
&\oplus(D^{(a-2,a-2)}\otimes D^{(1^2)}\otimes D^{(1^2)}),\\
D^{(a+1,a)}\downarrow_{\Sigma_{2a-3,2^2}} \cong &(D^{(a-1,a-2)}\otimes D^{(2)}\otimes D^{(2)})^2\oplus(D^{(a,a-3)}\otimes D^{(1^2)}\otimes D^{(2)})\\
&\oplus(D^{(a,a-3)}\otimes D^{(2)}\otimes D^{(2)})\oplus(D^{(a-1,a-2)}\otimes D^{(1^2)}\otimes D^{(2)})\\
&\oplus(D^{(a,a-3)}\otimes D^{(2)}\otimes D^{(1^2)})\oplus(D^{(a-1,a-2)}\otimes D^{(2)}\otimes D^{(1^2)})\\
&\oplus(D^{(a-1,a-2)}\otimes D^{(1^2)}\otimes D^{(1^2)}),\\
D^{(a+2,a)}\downarrow_{\Sigma_{2a-2,2^2}} \cong &(D^{(a,a-2)}\otimes D^{(2)}\otimes D^{(2)})^2\oplus(D^{(a-1,a-1)}\otimes D^{(1^2)}\otimes D^{(2)})\\
&\oplus(D^{(a-1,a-1)}\otimes D^{(2)}\otimes D^{(2)})\oplus(D^{(a,a-2)}\otimes D^{(1^2)}\otimes D^{(2)})\\
&\oplus(D^{(a-1,a-1)}\otimes D^{(2)}\otimes D^{(1^2)})\oplus(D^{(a,a-2)}\otimes D^{(2)}\otimes D^{(1^2)})\\
&\oplus(D^{(a,a-2)}\otimes D^{(1^2)}\otimes D^{(1^2)}),\\
D^{(a+3,a)}\downarrow_{\Sigma_{2a-1,2^2}} \cong &(D^{(a+1,a-2)}\otimes D^{(2)}\otimes D^{(2)})\oplus(D^{(a,a-1)}\otimes D^{(2)}\otimes D^{(2)})\\
&\oplus(D^{(a,a-1)}\otimes D^{(1^2)}\otimes D^{(2)})\oplus(D^{(a,a-1)}\otimes D^{(2)}\otimes D^{(1^2)})\\
&\oplus(D^{(a+1,a-2)}\otimes D^{(1^2)}\otimes D^{(1^2)}).
\end{aligned}$$

The only possible composition factors of $D^\lambda\downarrow_{\Sigma_4}$ are $D^{(4)}$, $D^{(3,1)}$ and $D^{(2^2)}$. So since $D^{(4)}\downarrow_{\Sigma_{2^2}} \cong D^{(2)}\otimes D^{(2)}$, $D^{(3,1)}\downarrow_{\Sigma_{2^2}} \cong (D^{(2)}\otimes D^{(2)})\oplus(D^{(2)}\otimes D^{(1^2)})\oplus(D^{(1^2)}\otimes D^{(2)})$ and $D^{(2^2)}\downarrow_{\Sigma_{2^2}} \cong (D^{(2)}\otimes D^{(2)})\oplus(D^{(1^2)}\otimes D^{(1^2)})$, the structure of $D^\lambda\downarrow_{\Sigma_{n-4,4}}$ follows. □

Lemma 5.2. *Let $p=5$ and $n\equiv\pm1 \mod 5$ with $n\geq 9$. If $\lambda=(a+b,a)\vdash n$ with $0\leq b\leq 3$ then there exists $\psi\in \mathrm{Hom}_{S_n}(M^{(n-3,1^3)},\mathrm{End}_F(D^\lambda))$ which does not vanish on $S^{(n-3,1^3)}$.*

Proof. It follows from Lemmas 4.3 and 5.1. □

Lemma 5.3. *Let $p=5$ and $n\equiv 0 \mod 5$ with $n\geq 9$. If $\lambda=(a+b,a)\vdash n$ with $0\leq b\leq 3$ then there exists $\psi\in \mathrm{Hom}_{S_n}(M^{(n-4,2^2)},\mathrm{End}_F(D^\lambda))$ which does not vanish on $S^{(n-4,2^2)}$.*

Proof. It follows from Lemmas 4.4 and 5.1. □

6 Mullineux fixed JS-partitions

Mullineux fixed JS-partitions also play a special role in the proof of Theorem 1.1 and so will be studied in this section.

Lemma 6.1. *Let $p \geq 3$ and $\lambda = \lambda^{\mathtt{M}} \vdash n$ be a JS-partition. Then $n \equiv h(\lambda)^2 \mod p$.*

Proof. Let $\lambda^0 := \lambda$ and then define recursively λ^i to be obtained from λ^{i-1} by removing the p-rim. From Theorem 4.1 of [9] we have that all the partitions λ^i are Mullineux fixed JS-partitions. Further if k is maximal such that $\lambda^k \neq ()$, then $\lambda^k = (1)$. In particular $|\lambda^k| \equiv h(\lambda^k)^2 \mod p$.

Assume that for a certain $1 \leq i \leq k$ we have that $|\lambda^i| \equiv h(\lambda^i)^2 \mod p$. From Theorem 4.1 of [9], one of the following holds:

(i) $|\lambda^{i-1}| - |\lambda^i| \equiv 2h(\lambda^i) + 1 \mod p$ and $h(\lambda^{i-1}) \equiv h(\lambda^i) + 1 \mod p$,

(ii) $|\lambda^{i-1}| - |\lambda^i| \equiv -2h(\lambda^i) + 1 \mod p$ and $h(\lambda^{i-1}) \equiv -h(\lambda^i) + 1 \mod p$,

(iii) $h(\lambda^i) \equiv 0 \mod p$, $|\lambda^{i-1}| - |\lambda^i| \equiv 0 \mod p$ and $h(\lambda^{i-1}) \equiv 0 \mod p$.

Using $|\lambda^i| \equiv h(\lambda^i)^2 \mod p$ it follows that in each of the above cases:

(i) $|\lambda^{i-1}| \equiv |\lambda^i| + 2h(\lambda^i) + 1 \equiv h(\lambda^i)^2 + 2h(\lambda^i) + 1 \equiv h(\lambda^{i-1})^2 \mod p$,

(ii) $|\lambda^{i-1}| \equiv |\lambda^i| - 2h(\lambda^i) + 1 \equiv h(\lambda^i)^2 - 2h(\lambda^i) + 1 \equiv h(\lambda^{i-1})^2 \mod p$,

(iii) $|\lambda^{i-1}| \equiv |\lambda^i| \equiv 0 \equiv h(\lambda^{i-1}) \mod p$.

The lemma then follows by induction. □

Lemma 6.2. *Let $p = 5$, $n \geq 5$ and $\lambda = \lambda^{\mathtt{M}} \vdash n$ be a JS-partition. Then there exists $i = \pm 1$ such that the following hold:*

- $D^\lambda{\downarrow}_{S_{n-1}} \cong D^{\tilde{e}_0(\lambda)}$,
- $\varepsilon_{\pm i}(\tilde{e}_0(\lambda)) = 1$, $\varepsilon_j(\tilde{e}_0(\lambda)) = 0$ *for* $j \neq \pm i$ *and*

$$D^\lambda{\downarrow}_{S_{n-2,2}} \cong (D^{\tilde{e}_i\tilde{e}_0(\lambda)} \otimes D^{(2)}) \oplus (D^{\tilde{e}_{-i}\tilde{e}_0(\lambda)} \otimes D^{(1^2)}),$$

- $\varepsilon_{-i}(\tilde{e}_i\tilde{e}_0(\lambda)), \varepsilon_{2i}(\tilde{e}_i\tilde{e}_0(\lambda)) = 1$, $\varepsilon_j(\tilde{e}_i\tilde{e}_0(\lambda)) = 0$ *for* $j \neq -i, 2i$. *Further* $\tilde{e}_{-i}\tilde{e}_i\tilde{e}_0(\lambda) = \tilde{e}_i\tilde{e}_{-i}\tilde{e}_0(\lambda)$ *and*

  $$D^\lambda{\downarrow}_{S_{n-3,3}} \cong (D^{\tilde{e}_{2i}\tilde{e}_i\tilde{e}_0(\lambda)} \otimes D^{(3)}) \oplus (D^{\tilde{e}_{-i}\tilde{e}_i\tilde{e}_0(\lambda)} \otimes A) \oplus (D^{\tilde{e}_{-2i}\tilde{e}_{-i}\tilde{e}_0(\lambda)} \otimes D^{(1^3)}),$$

  with $A \in \{D^{(2,1)}, D^{(3)} \oplus D^{(1^3)}\}$.

Proof. Notice that from Lemma 4.8 the unique normal node of λ has residue 0. So from Lemmas 2.3 and 2.4, $D^\lambda{\downarrow_{S_{n-1}}} \cong D^{\tilde{e}_0(\lambda)}$. From Proposition 3.6 of [23] we also have that

$$D^{\tilde{e}_0(\lambda)}{\downarrow_{S_{n-2}}} \cong D^\lambda{\downarrow_{S_{n-2}}} \cong D^\lambda{\downarrow_{S_{n-2}}} \cong D^\nu \oplus D^{\nu^{\mathtt{M}}}$$

with $\nu \neq \nu^{\mathtt{M}}$. From Lemmas 2.3 and 2.4 it then follows that there exist $i \neq k$ with $\varepsilon_i(\tilde{e}_0(\lambda)), \varepsilon_k(\tilde{e}_0(\lambda)) = 1$ and $\varepsilon_j(\tilde{e}_0(\lambda)) = 0$ else. From Lemma 4.8 we have that $\tilde{e}_0(\lambda) = \tilde{e}_0(\lambda)^{\mathtt{M}}$ and then $k = -i \neq 0$.

Let i be the residue of the top removable node of $\tilde{e}_0(\lambda)$ is normal. We will prove that $i = \pm 1$ and that $\varepsilon_{-i}(\tilde{e}_i\tilde{e}_0(\lambda)), \varepsilon_{2i}(\tilde{e}_i\tilde{e}_0(\lambda)) = 1$ and $\varepsilon_j(\tilde{e}_i\tilde{e}_0(\lambda)) = 0$ else. Further we will prove that $\tilde{e}_{-i}\tilde{e}_i\tilde{e}_0(\lambda) = \tilde{e}_i\tilde{e}_{-i}\tilde{e}_0(\lambda)$. Up to exchanging i and $-i$, this will prove the lemma, since $\lambda = \lambda^{\mathtt{M}}$, due to Lemmas 2.3, 2.4 and 4.8 and by comparing $D^\lambda{\downarrow_{S_{n-2,2}}}{\downarrow_{S_{n-3,1,2}}}$ and $D^\lambda{\downarrow_{S_{n-3,3}}}{\downarrow_{S_{n-3,1,2}}}$.

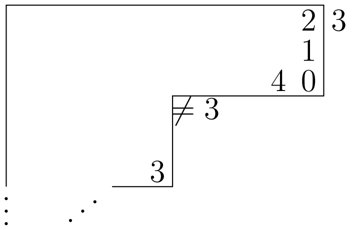
Assume that $\varepsilon_{-i}(\tilde{e}_i\tilde{e}_0(\lambda)) = 1$. Then $\varepsilon_i(\tilde{e}_{-i}\tilde{e}_0(\lambda)) = 1$ by Lemma 4.8. Let A and B be the normal node of $\tilde{e}_0(\lambda)$ of residue i and $-i$ respectively. Then, from Lemma 4.6, A is normal in $\tilde{e}_{-i}\tilde{e}_0(\lambda)$ of residue i and B is normal in $\tilde{e}_i\tilde{e}_0(\lambda)$ of residue $-i$. Since $\varepsilon_{\pm i}(\tilde{e}_0(\lambda)), \varepsilon_{\mp i}(\tilde{e}_{\pm i}\tilde{e}_0(\lambda)) = 1$, it follows that

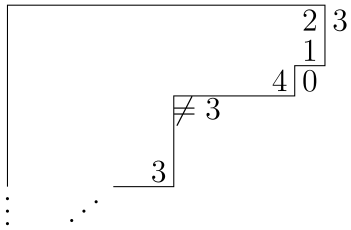
$$\tilde{e}_{-i}\tilde{e}_i\tilde{e}_0(\lambda) = \tilde{e}_{-i}(\tilde{e}_0(\lambda) \setminus A) = \tilde{e}_0(\lambda) \setminus \{A, B\} = \tilde{e}_i(\tilde{e}_0(\lambda) \setminus B) = \tilde{e}_i\tilde{e}_{-i}\tilde{e}_0(\lambda).$$

To prove the lemma it is then enough to prove that $i = \pm 1$ and that $\varepsilon_{-i}(\tilde{e}_i\tilde{e}_0(\lambda)), \varepsilon_{2i}(\tilde{e}_i\tilde{e}_0(\lambda)) = 1$ and $\varepsilon_j(\tilde{e}_i\tilde{e}_0(\lambda)) = 0$ else. From Lemma 1.8 of [23] we have that $h(\lambda) \geq 4$ and then from Lemma 2.2 of [3] that $\lambda_1 \geq \lambda_4 + 2$, as otherwise $\lambda_1^{\mathtt{M}} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 > \lambda_1$, contradicting $\lambda = \lambda^{\mathtt{M}}$.

Write $\lambda = (a_1^{b_1}, \ldots, a_h^{b_h})$ with $a_1 > \ldots > a_h \geq 1$ and $1 \leq b_j \leq 4$. From the previous part we have that $1 \leq b_1 \leq 3$ and that $h \geq 2$. Since λ is a JS-partition we have from Theorem D of [18] we have that $b_1 + b_2 + a_1 - a_2 \equiv 0$ mod 5. If $a_1 - a_2 = 1$ then we would have that $b_1 + b_2 = 4$, and then $\lambda_1 = a_1 = a_2 + 1 = \lambda_4$, leading to a contradiction. So $a_1 \geq a_2 + 2$. From Theorem D of [18] we also have that $(a_j^{b_j}, \ldots, a_h^{b_h})$ is a JS-partition for each $1 \leq j \leq h$. In particular if $\nu = (\psi_1, \ldots, \psi_l, a_j^{b_j}, \ldots, a_h^{b_h})$ is 5-regular with $\psi_l > a_j$ for some $1 \leq j \leq h$ and some $l \geq 1$, then the only possible normal nodes of ν are the removable nodes in the first l rows and the node $(l+b_j, a_j)$. This will be used in each of the following cases to find the normal nodes of $\tilde{e}_i\tilde{e}_0(\lambda)$.

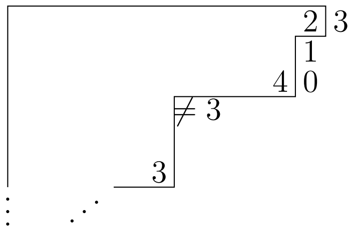
Assume first that $b_1 = 3$. Then, since λ is a JS-partition,

$\lambda =$ 

, $\tilde{e}_0(\lambda) =$ 

.

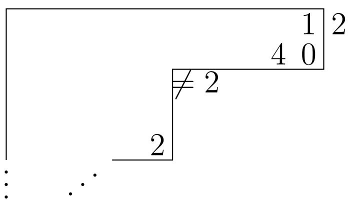
So in this case $i = 1$ and

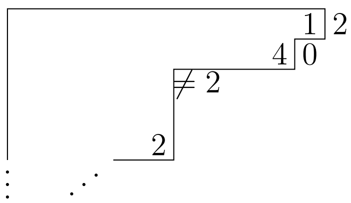
$\tilde{e}_i\tilde{e}_0(\lambda) =$ 

,

In particular $\tilde{e}_i\tilde{e}_0(\lambda) = (a_1, (a_1-1)^2, a_2^{b_2}, \ldots, a_h^{b_h})$ with $a_1 - 1 > a_2$. Since $(1, a_1)$ and $(3, a_1 - 1)$ are normal in $\tilde{e}_i\tilde{e}_0(\lambda)$ of residue 2 and 4 respectively while $(3 + b_2, a_2)$ is not normal, the lemma follows in this case.

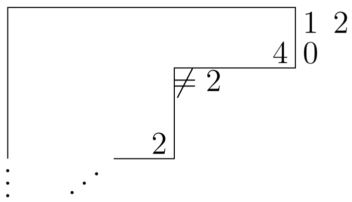
Assume next that $b_1 = 2$. Then, since λ is a JS-partition,

$\lambda =$ 

, $\tilde{e}_0(\lambda) =$ 

.

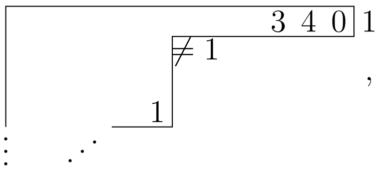
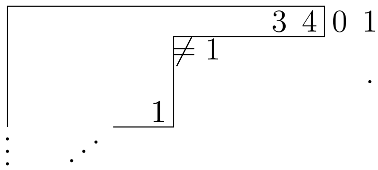
So in this case $i = 1$ and

$\tilde{e}_i\tilde{e}_0(\lambda) =$ 

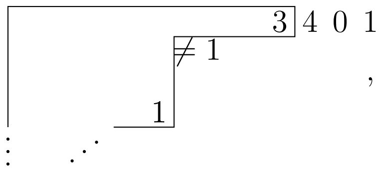
,

In particular $\tilde{e}_i\tilde{e}_0(\lambda) = ((a_1-1)^2, a_2^{b_2}, \ldots, a_h^{b_h})$ with $a_1 - 1 > a_2$. Since $(2, a_1 - 1)$ and $(2 + b_2, a_2)$ are normal in $\tilde{e}_i\tilde{e}_0(\lambda)$ of residue 4 and 2 respectively, the lemma follows in this case.

Assume now that $b_1 = 1$ and $a_1 \geq a_2 + 3$. Then, being λ a JS-partition,

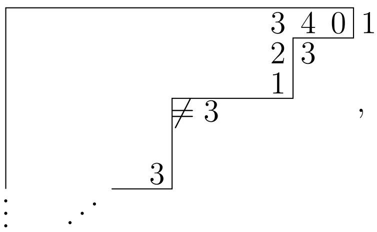
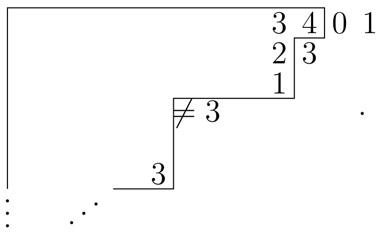
$\lambda =$  , $\tilde{e}_0(\lambda) =$  .

So in this case $i = 4$ and

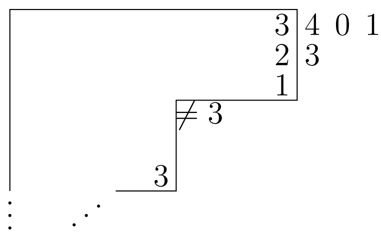
$\tilde{e}_i\tilde{e}_0(\lambda) =$  ,

In particular $\tilde{e}_i\tilde{e}_0(\lambda) = (a_1 - 2, a_2^{b_2}, \ldots, a_h^{b_h})$ with $a_1 - 2 > a_2$. Since $(1, a_1 - 2)$ and $(1 + b_2, a_2)$ are normal in $\tilde{e}_i\tilde{e}_0(\lambda)$ of residue 3 and 1 respectively, the lemma follows in this case.

Assume last that $b_1 = 1$ and $a_1 = a_2 + 2$. Then $b_2 = 2$ (from λ being a JS-partition) and $h \geq 3$ (as λ has at least 4 rows). So

$\lambda =$  , $\tilde{e}_0(\lambda) =$  .

In this case $i = 4$ and

$\tilde{e}_i\tilde{e}_0(\lambda) =$  .

Since $\tilde{e}_i\tilde{e}_0(\lambda) = (a_2^3, a_3^{b_3}, \ldots, a_h^{b_h})$ and the nodes $(3, a_2)$ and $(3 + b_3, a_3)$ are normal in $\tilde{e}_i\tilde{e}_0(\lambda)$ of residue 1 and 3 respectively the lemma follows also in this case. $\square$

Lemma 6.3. *Let $p = 5$ and $n \equiv \pm 1 \mod 5$ with $n \geq 6$. If $\lambda = \lambda^{\mathtt{M}} \vdash n$ then there exists $\psi \in \mathrm{Hom}_{S_n}(M^{(n-3,1^3)}, \mathrm{End}_F(D^\lambda))$ which does not vanish on $S^{(n-3,1^3)}$.*

Proof. It follows from Lemmas 4.3 and 6.2. $\square$

7 Split-non-split case

In this section we will prove Theorem 1.1 in the case where one of the two irreducible A_n-modules D_1, D_2 splits when reduced to A_n, while the other doesn't.

Lemma 7.1. *Let $p \geq 3$ and $\lambda, \mu \vdash n$ be p-regular. If $\lambda = \lambda^{\mathtt{M}}$, $\mu \neq \mu^{\mathtt{M}}$ and $E^\lambda_\pm \otimes E^\mu$ is irreducible then $D^\lambda \otimes D^\mu \cong D^\nu \oplus D^{\nu^{\mathtt{M}}}$ for some $\nu \neq \nu^{\mathtt{M}}$. In particular*

$$\dim \mathrm{Hom}_{S_n}(\mathrm{End}_F(D^\lambda), \mathrm{End}_F(D^\mu)) = 2.$$

Proof. See Lemma 3.1 of [7]. □

Theorem 7.2. *Let $p \geq 3$, $n \geq 6$ and $\lambda, \mu \vdash n$ be p-regular. If $\lambda = \lambda^{\mathtt{M}}$, $\mu \neq \mu^{\mathtt{M}}$, $E^\lambda_\pm$ and E^μ are not 1-dimensional and $E^\lambda_\pm \otimes E^\mu$ is irreducible then λ is a JS-partition.*

Proof. From Lemmas 3.1, 4.1, 4.5 there exist $\psi_\mu : M^{(n-2,2)} \to \mathrm{End}_F(D^\lambda)$ which does not vanish on $S^{(n-2,2)}$.

If λ is not a JS-partition, from Lemmas 3.1, 4.1, 4.9, 4.11 and 4.12 there exist $\psi_{\lambda,1}, \psi_{\lambda,2} : M^{(n-2,2)} \to \mathrm{End}_F(D^\lambda)$ such that any non-zero linear combination of $\psi_{\lambda,1}$ and $\psi_{\lambda,2}$ does not vanish on $S^{(n-2,2)}$.

So from Lemma 4.2 it follows that

$$\dim \mathrm{Hom}_{\Sigma_n}(\mathrm{End}_F(D^\lambda), \mathrm{End}_F(D^\mu)) \geq 3$$

if λ is not a JS-partition ($\mathrm{End}_F(D^\lambda)$ and $\mathrm{End}_F(D^\mu)$ also have a quotient and a submodule isomorphic to $D^{(n)}$). The lemma then follows by Lemma 7.1 □

Theorem 7.3. *Let $p = 5$. Let λ, μ be 5-regular partitions with $\lambda = \lambda^{\mathtt{M}}$ and $\mu \neq \mu^{\mathtt{M}}$ such that $E^\lambda_\pm$ and E^μ are not 1-dimensional. If $E^\lambda_\pm \otimes E^\mu$ is irreducible then λ is a JS-partition and $\mu \in \{(n-1,1), (n-1,1)^{\mathtt{M}}\}$.*

Proof. For $n \leq 7$ the lemma follows by considering each case separately. So we can assume that $n \geq 8$. By Theorem 7.2 we have that λ is a JS-partition. So $n \equiv 0, 1$ or $4 \mod 5$ by Lemma 6.1. From Lemma 1.8 of [23] we have that $h(\lambda) \geq 4$. Further from Lemmas 3.1, 4.1 and 4.5 there exist $\psi_{\lambda,2} : M^{(n-2,2)} \to \mathrm{End}_F(D^\lambda)$ and $\psi_{\mu,2} : M^{(n-2,2)} \to \mathrm{End}_F(D^\mu)$ which do not vanish on $S^{(n-2,2)}$.

If $\mu, \mu^{\mathtt{M}} \neq (n-k, k)$ with $k = 1$ or $n - 2k \leq 3$ then, from Corollaries 3.9 and 4.12 of [10] and Lemma 3.1, for some $j \in \{3, 4\}$ there exist $\psi_{\lambda,j}$:

$M^{(n-j,j)} \to \mathrm{End}_F(D^\lambda)$ and $\psi_{\mu,j} : M^{(n-j,j)} \to \mathrm{End}_F(D^\mu)$ which do not vanish on $S^{(n-j,j)}$.

If $\mu, \mu^{\mathtt{M}} = (n-k,k)$ with $n-2k \leq 3$ and $n \equiv 0 \mod 5$ then there exist $\psi_{\lambda,2^2} : M^{(n-4,2^2)} \to \mathrm{End}_F(D^\lambda)$ and $\psi_{\mu,2^2} : M^{(n-4,2^2)} \to \mathrm{End}_F(D^\mu)$ which do not vanish on $S^{(n-4,2^2)}$ by Lemmas 3.6 and 5.3.

If $\mu, \mu^{\mathtt{M}} = (n-k,k)$ with $n-2k \leq 3$ and $n \equiv \pm 1 \mod 5$ then there exist $\psi_{\lambda,1^3} : M^{(n-3,1^3)} \to \mathrm{End}_F(D^\lambda)$ and $\psi_{\mu,1^3} : M^{(n-3,1^3)} \to \mathrm{End}_F(D^\mu)$ which do not vanish on $S^{(n-3,1^3)}$ by Lemmas 5.2 and 6.3.

In either of these cases it follows from Lemma 4.2 that

$$\dim \mathrm{Hom}_{\Sigma_n}(\mathrm{End}_F(D^\lambda), \mathrm{End}_F(D^\mu)) \geq 3$$

and so from Lemma 7.1 that $E^\lambda \pm \otimes E^\mu$ is not irreducible. □

Theorem 7.4. *Let $p \geq 3$ and λ be a p-regular partitions with $\lambda = \lambda^{\mathtt{M}}$. Then $E^\lambda_\pm \otimes E^{(n-1,1)}$ is irreducible if and only if $n \not\equiv 0 \mod p$ and λ is a JS-partition. In this case, if ν is obtained from λ by removing the top removable node and adding the bottom addable node, then $E^\lambda_\pm \otimes E^{(n-1,1)} \cong E^\nu$.*

Proof. See Theorem 3.3 of [7] and Lemma 6.1. □

8 Double-split case

In this section we will prove Theorem 1.1 in the case where both irreducible A_n-modules D_1, D_2 split when reduced to A_n.

Lemma 8.1. *Let λ, μ be p-regular partitions with $\lambda = \lambda^{\mathtt{M}}$ and $\mu = \mu^{\mathtt{M}}$. Also let $\varepsilon_1, \varepsilon_2 \in \{\pm\}$. If $E^\lambda_{\varepsilon_1} \otimes E^\mu_{\varepsilon_2}$ is irreducible then*

$$\dim \mathrm{Hom}_{A_n}(\mathrm{Hom}_F(E^\lambda_{\varepsilon_1}, E^\lambda_{\delta_1}), \mathrm{Hom}_F(E^\mu_{\delta_2}, E^\mu_{\varepsilon_2})) \leq 1$$

for any combination $\delta_1, \delta_2 \in \{\pm\}$.

Proof. See Lemma 3.4 of [7] (and its proof). □

Lemma 8.2. *Let $p \geq 3$ and $n \geq 4$. Let λ, μ be p-regular partitions with $\lambda = \lambda^{\mathtt{M}}$ and $\mu = \mu^{\mathtt{M}}$. Assume that $E^\lambda_{\varepsilon_1} \otimes E^\mu_{\varepsilon_2}$ is irreducible for some $\varepsilon_1, \varepsilon_2 \in \{\pm\}$. Then, up to exchange of λ and μ,*

$$\begin{aligned} \dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) &= \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}}) + 1, \\ \dim \mathrm{End}_{S_{n-2,2}}(D^\mu{\downarrow}_{S_{n-2,2}}) &\leq \dim \mathrm{End}_{S_{n-1}}(D^\mu{\downarrow}_{S_{n-1}}) + 2. \end{aligned}$$

Proof. Notice first that $(n) > (n)^{\mathtt{M}}$ and $(n-2,2) > (n-2,2)^{\mathtt{M}}$ (this follows from Lemma 1.8 of [23] and from $n \geq 4$, so that $(n-2,2)^{\mathtt{M}} \neq (n)$).

From Lemmas 4.1 and 4.13

$$\begin{aligned}\dim \mathrm{End}_{S_\alpha}(D^\lambda{\downarrow}_{S_\alpha}) &= \dim \mathrm{Hom}_{S_n}(M^\alpha, \mathrm{End}_F(D^\lambda))\\ &= \dim \mathrm{Hom}_{A_n}(M^\alpha, \mathrm{Hom}_F(E^\lambda_+ \oplus E^\lambda_-, E^\lambda_{\varepsilon_1}))\end{aligned}$$

and similarly for μ.

From Lemma 4.5 we have that

$$\begin{aligned}\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) &\geq \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}}) + 1,\\ \dim \mathrm{End}_{S_{n-2,2}}(D^\mu{\downarrow}_{S_{n-2,2}}) &\geq \dim \mathrm{End}_{S_{n-1}}(D^\mu{\downarrow}_{S_{n-1}}) + 1.\end{aligned}$$

Assume first that

$$\begin{aligned}\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) &\geq \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}}) + 2,\\ \dim \mathrm{End}_{S_{n-2,2}}(D^\mu{\downarrow}_{S_{n-2,2}}) &\geq \dim \mathrm{End}_{S_{n-1}}(D^\mu{\downarrow}_{S_{n-1}}) + 2.\end{aligned}$$

Then, from Lemmas 3.1 and 4.2, we have that

$$\dim \mathrm{Hom}_{A_n}(\mathrm{Hom}_F(E^\lambda_{\varepsilon_1}, E^\lambda_+ \oplus E^\lambda_-), \mathrm{Hom}_F(E^\mu_+ \oplus E^\mu_-, E^\lambda_{\varepsilon_2}) \geq 1 + 2 \cdot 2 = 5,$$

contradicting that $E^\lambda_{\varepsilon_1} \otimes E^\mu_{\varepsilon_2}$ is irreducible, due to Lemma 8.1.

Up to exchange we can then assume that

$$\begin{aligned}\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) &= \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}}) + 1,\\ \dim \mathrm{End}_{S_{n-2,2}}(D^\mu{\downarrow}_{S_{n-2,2}}) &\geq \dim \mathrm{End}_{S_{n-1}}(D^\mu{\downarrow}_{S_{n-1}}) + 3.\end{aligned}$$

Then, from Lemma 4.1 and by self-duality of $M^{(n-1,1)}$ and $M^{(n-2,2)}$,

$$\begin{aligned}&\dim \mathrm{Hom}_{A_n}(\mathrm{Hom}_F(E^\lambda_{\varepsilon_1}, E^\lambda_+ \oplus E^\lambda_-), M^{(n-2,2)})\\ &\quad = \dim \mathrm{Hom}_{A_n}(\mathrm{Hom}_F(E^\lambda_{\varepsilon_1}, E^\lambda_+ \oplus E^\lambda_-), M^{(n-1,1)}) + 1\end{aligned}$$

and

$$\begin{aligned}&\dim \mathrm{Hom}_{A_n}(M^{(n-2,2)}, \mathrm{Hom}_F(E^\mu_+ \oplus E^\mu_-, E^\mu_{\varepsilon_2}))\\ &\quad \geq \dim \mathrm{Hom}_{A_n}(M^{(n-1,1)}, \mathrm{Hom}_F(E^\mu_+ \oplus E^\mu_-, E^\lambda_{\varepsilon_2})) + 3.\end{aligned}$$

In particular there exist $\delta_1, \delta_2 \in \{\pm\}$ with

$$\begin{aligned}&\dim \mathrm{Hom}_{A_n}(\mathrm{Hom}_F(E^\lambda_{\delta_1}, E^\lambda_{\varepsilon_1}), M^{(n-2,2)})\\ &\quad \geq \dim \mathrm{Hom}_{A_n}(\mathrm{Hom}_F(E^\lambda_{\delta_1}, E^\lambda_{\varepsilon_1}), M^{(n-1,1)}) + 1\end{aligned}$$

and

$$\begin{aligned}&\dim \mathrm{Hom}_{A_n}(M^{(n-2,2)}, \mathrm{Hom}_F(E^\mu_{\delta_2}, E^\mu_{\varepsilon_2}))\\ &\quad\geq \dim \mathrm{Hom}_{A_n}(M^{(n-1,1)}, \mathrm{Hom}_F(E^\mu_{\delta_2}, E^\lambda_{\varepsilon_2})) + 2.\end{aligned}$$

From Lemmas 2.1, 3.1 and 4.2 it then follows that

$$\dim \mathrm{Hom}_{A_n}(\mathrm{Hom}_F(E^\lambda_{\varepsilon_1}, E^\lambda_{\delta_1}), \mathrm{Hom}_F(E^\mu_{\delta_2}, E^\lambda_{\varepsilon_2}) \geq 0 + 2,$$

again contradicting that $E^\lambda_{\varepsilon_1} \otimes E^\mu_{\varepsilon_2}$ is irreducible, due to Lemma 8.1. □

Theorem 8.3. *Let $p = 5$. If $\lambda, \mu \vdash n$ are 5-regular partitions with $\lambda = \lambda^{\mathtt{M}}$ and $\mu = \mu^{\mathtt{M}}$ then $E^\lambda_{\varepsilon_1} \otimes E^\mu_{\varepsilon_2}$ is not irreducible for any choice of $\varepsilon_1, \varepsilon_2 \in \{\pm\}$, unless $n \leq 4$ in which case $E^\lambda_\pm$ and $E^\mu_\pm$ are 1-dimensional.*

Proof. For $n \leq 7$ the lemma can be proved by considering each case separately.

So we can assume that $n \geq 8$. Notice first that $(n-a, a) > (n-a, a)^{\mathtt{M}}$ for $0 \leq a \leq 4$ (this follows from Lemma 1.8 of [23] and from $n \geq 8$, so that $h((n)^{\mathtt{M}}) = 4$

From Lemma 1.8 of [23] we have that $h(\lambda), h(\mu) \geq 4$. So, from Corollary 3.9 of [10],

$$\begin{aligned}
\dim \mathrm{Hom}_{S_n}(M^{(n-3,3)}, \mathrm{End}_F(D^\lambda)) &> \dim \mathrm{Hom}_{S_n}(M^{(n-2,2)}, \mathrm{End}_F(D^\lambda)),\\
\dim \mathrm{Hom}_{S_n}(M^{(n-4,4)}, \mathrm{End}_F(D^\lambda)) &> \dim \mathrm{Hom}_{S_n}(M^{(n-3,3)}, \mathrm{End}_F(D^\lambda)),\\
\dim \mathrm{Hom}_{S_n}(M^{(n-3,3)}, \mathrm{End}_F(D^\mu)) &> \dim \mathrm{Hom}_{S_n}(M^{(n-2,2)}, \mathrm{End}_F(D^\mu)),\\
\dim \mathrm{Hom}_{S_n}(M^{(n-4,4)}, \mathrm{End}_F(D^\mu)) &> \dim \mathrm{Hom}_{S_n}(M^{(n-3,3)}, \mathrm{End}_F(D^\mu)).
\end{aligned}$$

From Lemma 8.2 we can assume that

$$\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda{\downarrow}_{S_{n-2,2}}) = \dim \mathrm{End}_{S_{n-1}}(D^\lambda{\downarrow}_{S_{n-1}}) + 1.$$

Assume first that

$$\dim \mathrm{End}_{S_{n-2,2}}(D^\mu{\downarrow}_{S_{n-2,2}}) > \dim \mathrm{End}_{S_{n-1}}(D^\mu{\downarrow}_{S_{n-1}}) + 1.$$

Then from Lemmas 3.1, 4.1, 4.2 and 4.13 we have that

$$\dim \mathrm{Hom}_{S_n}(\mathrm{Hom}_F(E^\lambda_{\varepsilon_1}, E^\lambda_+ \oplus E^\lambda_-), \mathrm{Hom}_F(E^\mu_+ \oplus E^\mu_-, E^\mu_{\varepsilon_2})) \geq 1+0+2+1+1 = 5.$$

In particular, from Lemma 8.1, $E^\lambda_\pm \otimes E^\mu_\pm$ is not irreducible.

So we may now assume (from Lemma 4.5) that

$$\dim \mathrm{End}_{S_{n-2,2}}(D^\lambda\downarrow_{S_{n-2,2}}) = \dim \mathrm{End}_{S_{n-1}}(D^\lambda\downarrow_{S_{n-1}}) + 1,$$
$$\dim \mathrm{End}_{S_{n-2,2}}(D^\mu\downarrow_{S_{n-2,2}}) = \dim \mathrm{End}_{S_{n-1}}(D^\mu\downarrow_{S_{n-1}}) + 1.$$

From Lemmas 4.9, 4.11 and 4.12 we then have that λ and μ are JS-partitions.

From Lemma 6.2 we have that $(E^\lambda_+ \oplus E^\lambda_-)\downarrow_{A_{n-2,2}} \cong D^\lambda\downarrow_{A_{n-2,2}}$ has only 2 composition factors (since so does $D^\lambda\downarrow_{S_{n-2,2}}$ and none of these composition factors is fixed under tensoring with sign). In particular $E^\lambda_{\varepsilon_1}\downarrow_{A_{n-2,2}}$ is simple. From Lemma 1.1 of [7] and from Lemma 6.2 we have that $(E^\lambda_+ \oplus E^\lambda_-)\downarrow_{A_{n-3,3}} \cong D^\lambda\downarrow_{A_{n-3,3}}$ is semisimple and has at least 3 composition factors. In particular $E^\lambda_{\varepsilon_1}\downarrow_{A_{n-3,3}}$ is semisimple with at least 2 composition factors. So

$$\dim \mathrm{End}_{A_{n-3,3}}(E^\lambda_{\varepsilon_1}\downarrow_{A_{n-3,3}}) > \dim \mathrm{End}_{A_{n-2,2}}(E^\lambda_{\varepsilon_1}\downarrow_{A_{n-2,2}}).$$

Similarly

$$\dim \mathrm{End}_{A_{n-3,3}}(E^\mu_{\varepsilon_2}\downarrow_{A_{n-3,3}}) > \dim \mathrm{End}_{A_{n-2,2}}(E^\mu_{\varepsilon_2}\downarrow_{A_{n-2,2}}).$$

From Lemmas 2.1, 3.1 and 4.2 it then follows that

$$\dim \mathrm{End}_{A_n}(E^\lambda_{\varepsilon_1} \otimes E^\mu_{\varepsilon_2}) = \dim \mathrm{Hom}_{A_n}(\mathrm{End}_F(E^\lambda_{\varepsilon_1}), \mathrm{End}_F(E^\mu_{\varepsilon_2})) \geq 1+0+1 = 2,$$

so that again $E^\lambda_{\varepsilon_1} \otimes E^\mu_{\varepsilon_2}$ is not irreducible. $\square$

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