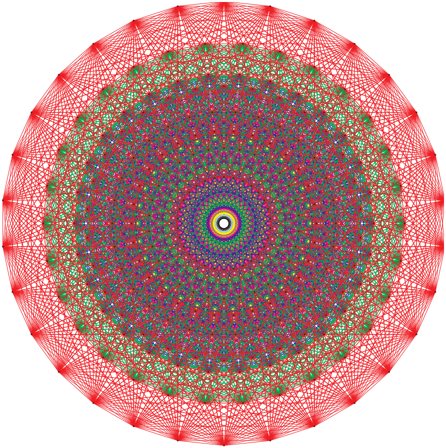
How E_8 made the headlines

Max Neunhöffer



A news story

On 20 March 2007 The New York Times wrote:

The Scientific Promise of Perfect Symmetry

It is one of the most symmetrical mathematical structures in the universe. It may underlie the Theory of Everything that physicists seek to describe the universe.

Other quotes:

- *248-dimension maths puzzle solved*
- *The "structure" is described in the form of a vast matrix.*
- *Mapping the 248-dimensional structure, called E8, took four years of work and produced more data than the Human Genome Project, researchers said.*

Classification

The unitary representations of the split real form of the simple complex Lie group E_8 have been classified.

What is a Lie group?

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

A Lie group over $\mathbb{K}$ is both

- a smooth $\mathbb{K}$-manifold, and
- a group with smooth multiplication and inversion.

(imagine: a closed subgroup of some $\mathrm{GL}_n(\mathbb{K})$)

Examples:

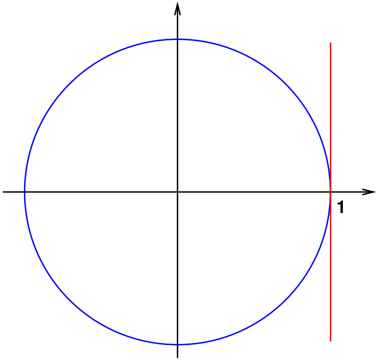
$\mathrm{GL}_n(\mathbb{C}) := \{A \in \mathbb{C}^{n\times n} \mid \det A \neq 0\}$	"general linear"
$\mathrm{GL}_n(\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}$	"general linear"
$\mathrm{SL}_n(\mathbb{K}) := \{A \in \mathbb{K}^{n\times n} \mid \det A = 1\}$	"special linear"
$\mathrm{U}_n := \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A \cdot \overline{A^T} = \mathbf{1}\}$	"unitary"
$\mathrm{SU}_n := \mathrm{U}_n \cap \mathrm{SL}_n(\mathbb{C})$	"special unitary"
$\mathrm{O}_n := \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A \cdot A^T = \mathbf{1}\}$	"orthogonal"
$\mathrm{SO}_n := \mathrm{O}_n \cap \mathrm{SL}_n(\mathbb{C})$	"special orthogonal"

The easiest example

Let's look a bit closer at

$$\mathrm{SO}_2 = \left\{ \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \middle| 0 \leq \alpha < 2\pi \right\} < \mathrm{GL}_2(\mathbb{R}).$$

It is S^1: Identify points on the sphere with rotations of the plane around the origin.



We see that it is smooth and compact (closed+bounded)!

What does E_8 mean?

Theorem (Lie correspondence)

A connected Lie group is (up to a discrete, central extension) determined by its Lie algebra, which is the tangent space at the identity.

Simple complex Lie algebras are classified by the Dynkin type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 or G_2 (Killing, 1887).

Classification

This leads to a classification of the connected simple complex Lie groups.

E_8 is simply the largest of the 5 exceptional cases.

Real and complex

Example: The real and complex worlds are related!

Look at $G := \mathrm{SL}_n(\mathbb{C})$ and two automorphisms of order 2:

Automorphisms:	$c : A \mapsto \overline{A}$	$t : A \mapsto \overline{A^{-T}}$
Invariants:	$G^c = \mathrm{SL}_n(\mathbb{R})$	$G^t = \mathrm{SU}_n$
Topology:	not compact	compact
Name:	"split"	compact

Def: $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SU}(n)$ are real forms of $\mathrm{SL}_n(\mathbb{C})$.
$\mathrm{SL}_n(\mathbb{C})$ is the complexification of both.

Theorem (Cartan, 1890)

A connected, semisimple, complex Lie group has only finitely many real forms.
There is always a split one which is not compact.

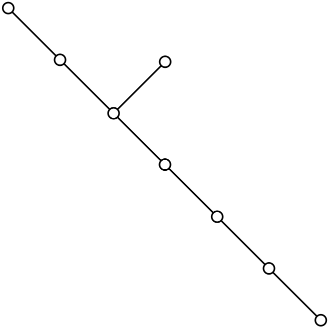
The split real form $E_8{}^8$ of E_8 ...

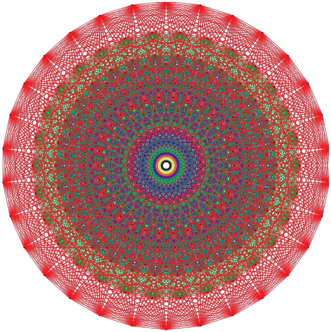
... is a 248-dimensional real non-compact Lie group.

E_8 is combinatorially described by its root system in $\mathbb{R}^8$:

$$\Phi \quad := \quad \{a \in \{\pm\tfrac{1}{2}\}^8 \mid \textstyle\sum a_i \text{ even}\} \cup \{a \in \{0, \pm 1\}^8 \mid \textstyle\sum (a_i)^2 = 2\} \quad \subseteq \mathbb{R}^8$$

$\langle\Phi\rangle_{\mathbb{Z}}$ is the E_8-lattice: $\left\{a \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 \mid \sum a_i \text{ even}\right\}$





$E_8{}^8$ is a group of symmetries of a 57-dimensional variety.

Representations

Definition (Representation)

A representation of a Lie group G is a continuous group homomorphism $D : G \to \mathrm{GL}(V)$, (V a $\mathbb{C}$-vector space).

irreducible $:\Longleftrightarrow$ only $\{0\}$ and V are $D(G)$-invariant closed subspaces

unitary $:\Longleftrightarrow$ $D(G) \subseteq U(V) < \mathrm{GL}(V)$ (for this V must be a Hilbert space)

Fact

An irreducible representation $D : G \to \mathrm{GL}_n(\mathbb{C})$ is determined up to equivalence by its character:

$$\chi_D : G \to \mathbb{C}, \quad g \mapsto \mathrm{Tr}(D(g)).$$

So characters are functions on G which are constant on conjugacy classes.

Def: $\mathrm{Irr}(G) := \{\text{characters of irreducible representations}\}$

The Atlas project

Aims

Collect and make available information on semisimple Lie groups and their (unitary) representations.

http://www.liegroups.org/

People: The core group

Jeffrey Adams	(University of Maryland),
Dan Barbasch	(Cornell),
John Stembridge	(University of Michigan),
Peter Trapa	(University of Utah),
Marc van Leeuwen	(Poitiers),
David Vogan	(MIT), and
Fokko du Cloux	(Lyons, until his death in 2006).

Questions about character tables

Fundamental questions

1. How can a character be described in a finite way, if G has infinitely many conjugacy classes?
2. How can a finite computation or data collection describe infinitely many, possibly infinite-dimensional irreducible unitary representations?

Good news about compact Lie groups:

- Every finite-dimensional, irreducible representation is equivalent to a unitary one.
- Every unitary representation is a direct sum of finite-dimensional irreducible ones (Peter and Weyl).

Maximal Tori

Definition (Torus)

Let $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. A torus is a direct product

$$\mathbb{T} := \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{r \text{ factors}}$$

This is a compact, abelian Lie group,
with set of characters $\{\chi_a(s) = \prod_{i=1}^r s_i^{a_i} \mid a \in \mathbb{Z}^r\}$.

Theorem (Maximal Tori)

Let G be a connected, compact Lie group. Then G has a maximal torus $\mathbb{T}$
The normaliser $N_G(\mathbb{T})$ is a closed subgroup and $N_G(\mathbb{T})/\mathbb{T}$ is a finite group, the Weyl group W.
Every conjugacy class of G meets $\mathbb{T}$.
$x, y \in \mathbb{T}$ conjugate in $G \iff x, y$ conjugate in $N_G(\mathbb{T})$

Character tables for compact Lie groups

Theorem (Weyl Character Formula)

G: connected, compact Lie group, $\mathbb{T}$: a maximal torus.

$$\mathrm{Irr}(G) \stackrel{1-1}{\longleftrightarrow} \Big\{ \underbrace{\text{“highest weights”}}_{\text{come from the root system}} \Big\}$$

Each character is given explicitly as a quotient of two finite integral linear combinations of characters of $\mathbb{T}$.

$$G = SU_2 = \left\{ \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix} \in \mathbb{C}^{2\times 2} \;\middle|\; |z|^2 + |w|^2 = 1 \right\}$$

$$\mathbb{T} = S^1 = \left\{ \begin{bmatrix} z & 0 \\ 0 & \overline{z} \end{bmatrix} \in \mathbb{C}^{2\times 2} \;\middle|\; |z|^2 = 1 \right\}$$

The characters of $\mathbb{T}$ are indexed by $\mathbb{Z}$, the highest weights are $\mathbb{N} \cup \{0\}$, the character χ_n on $\mathbb{T}$ is

$$\chi_n(z) = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}.$$

Trouble ahead

- There are irreducible representations which are not equivalent to unitary ones.
- There are infinite-dimensional, unitary, irreducible representations.
- The operators $D(g) \in \mathrm{GL}(V)$ do not necessarily have a trace, if $\dim(V) = \infty$!
  Characters are now distributions (Harish-Chandra).

Theorem (Langlands, Knapp-Zuckerman)

Let G be a reductive, algebraic Lie group. Then the irreducible characters are classified.

Big questions:

Which of them are unitary and what are their dimensions and character values?

"Character tables" of non-compact Lie groups

Theorem (Jantzen-Zuckerman "translation principle")

$$\mathrm{Irr}(G) = \dot{\bigcup}_{i=1}^{N} \mathcal{F}_i \qquad (\text{"translation families"})$$

Each $\mathcal{F}_i$ is parametrised by the set $\mathrm{IChar}(G)$ of infinitesimal characters: $\mathcal{F}_i = \{\chi_i^\lambda \mid \lambda \in \mathrm{IChar}(G)\}$.

$$\chi_i^\lambda = \sum_{j=1}^{N} P_{i,j}(1) \cdot \Theta_j^\lambda \qquad \text{for } 1 \le i \le N$$

for certain functions $\Theta_1^\lambda, \ldots, \Theta_N^\lambda$ (Harish-Chandra).

The polynomials $P_{i,j} \in \mathbb{Z}[q, q^{-1}]$ are the famous Kazhdan-Lusztig-Vogan-polynomials.

The $P_{i,j}$ do not depend on λ, and $P_{i,j} = 0$ for $i < j$.

$\rightarrow$ Very deep results about intersection homology by Kazhdan, Lusztig, Beilinson, Bernstein, Vogan, ...

The ${E_8}^8$-computation

Facts:

- For ${E_8}^8$ there are 453060 translation families.
  $\Longrightarrow 453060^2/2$ polynomials, degree up to 31.
- They can be computed by the Kazhdan-Lusztig-Algorithm.
- No bound for the integer coefficients was known!

Size of the computation:

- At first sight, 480 GB of RAM seemed to be needed!
- This was reduced to about 150 GB by various tricks.
- Finally, computing modulo 251, 253, 255 and 256 separately did the job.
- Altogether, the computation now needs about 50 hours and produces 60 GB of output.

How to explain this to the public?

Recipe:

- The idea was to promote mathematics in public.
- Original plan: reach science and technology media, like "Science"
- Press release by AIM and researchers involved
- Professional public relations help was hired.
- It was a lot of work, with long preparations.

Ingredients:

- Needed: "hooks" to be understood by the public: Symmetry, size, team effort, implications for research
- Critically important: a good accompanying web page
- Necessary: a good picture
- Good: associate news release with an event
- Extremely valuable: quotes from external experts

Lesson learned:

With enough effort, it can be done!

Thanks!