Algorithmic Generalisations of Small Cancellation Theory

Max Neunhöffer

joint work with Jeffrey Burdges, Stephen Linton,

Richard Parker and Colva Roney-Dougal

Aberdeen, 29 November 2012

Question

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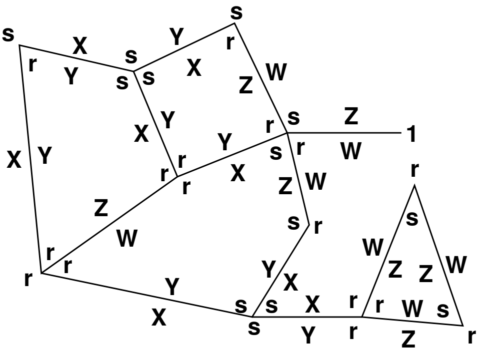
Can we solve the word problem?

We draw connected finite graphs in the plane and label them:



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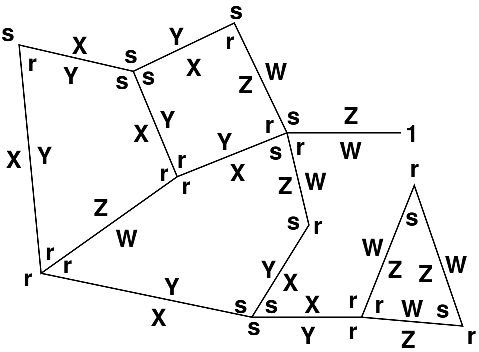

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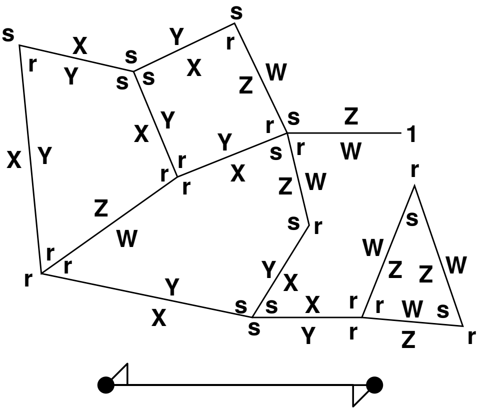

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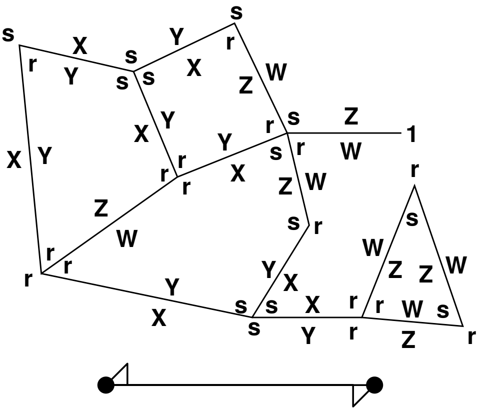

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Each half-edge is labelled at the start vertex and along the half-edge.

Diagram problems

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Problem (Diagram boundary problem)

Algorithmically devise a procedure that decides for any cyclic word w, whether or not there is a diagram such that

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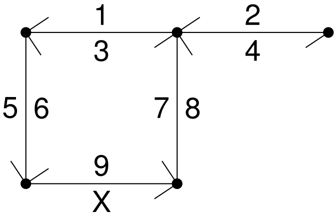
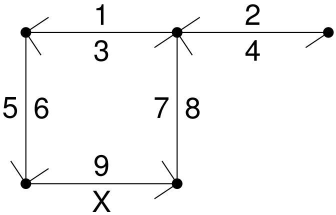

1
2
3
4
5
6
7
8
9
X



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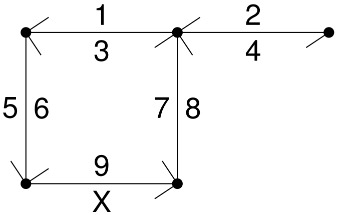

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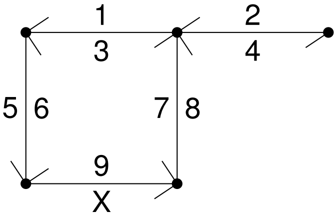

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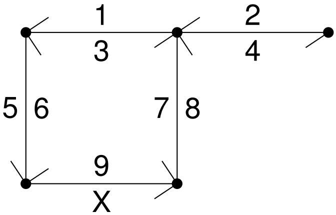

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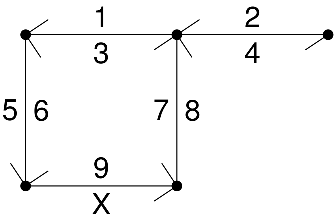
Lemma (Graph equivalence)

Finite connected planar embedded graphs with $n/2$ edges



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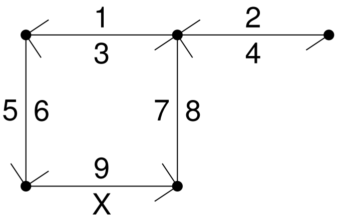

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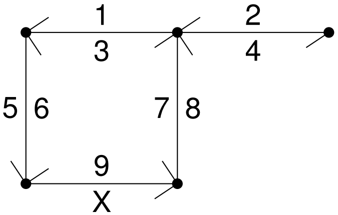

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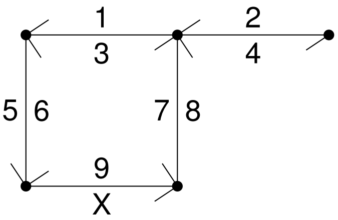

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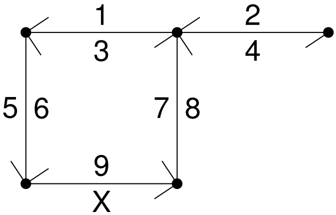

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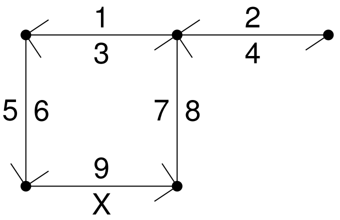

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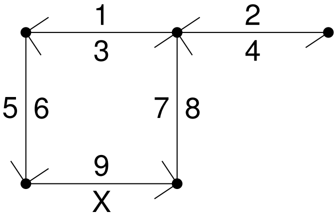

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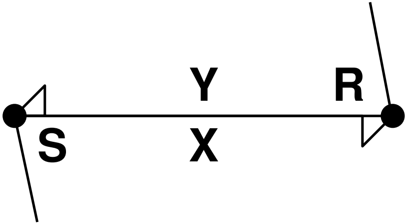
Rules for the labels

We label every half-edge with two symbols,

- one for the corner to the right of where it starts, and
- one for the right hand side of it:



We now need rules for the corner labels and the edge labels.

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A pongo is a set P with a subset $P_+ \subset P$, such that $P_0 := P \dot{\cup} \{0\}$ is a semigroup with 0 and:

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The elements in P_+ are called acceptors.

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Using a finite pongo is equivalent to using a finite state automaton.

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(For the experts:
This is a generalisation of the rules of van Kampen diagrams.)

Definition (Valid diagram)

Let P be a pongo and A be an edge alphabet. A valid diagram is: an $n \in \mathbb{N}$ and three permutations $E, F, V \in S_n$ and a labelling function $\ell : \{1, \ldots, n\} \to P \times A, x \mapsto (\ell_P(x), \ell_A(x))$, such that

- $EFV = 1$,
- E is a fixed point free involution,
- $\langle E, F \rangle$ is a transitive subgroup of S_n,
- the total number of cycles in E, F and V is $n + 2$,
- $\ell_P(x) \cdot \ell_P(xV) \cdot \ell_P(xV^2) \cdot \cdots \in P_+$ for every V-cycle $x \langle V \rangle$, and
- $\ell_A(xE) = \overline{\ell_A(x)}$ for all E-cycles (x, xE).

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Problem (Diagram boundary problem)

Algorithmically devise a procedure that decides for any cyclic word w in $P \times A$, whether or not there is a valid diagram such that

- *every internal F-cycle is labelled by a relator, and*
- *the external F-cycle is labelled by w.*

Problem (Isoperimetric inequality)

Algorithmically find and prove a function $\mathcal{D} : \mathbb{N} \to \mathbb{N}$, such that for every cyclic word $w \in P \times A$ of length k that is the boundary label of a valid diagram, there is one with at most $\mathcal{D}(k)$ internal F-cycles.

If there is a linear $\mathcal{D}$, we call (P, A, R) hyperbolic.

$G := \langle S, R, T \mid SR, T^2, S^3, (ST)^7, (STS^2T)^{13} \rangle$ can be studied by:

$$\begin{aligned}
P &= \{S, R, 1\} \text{ with } P_+ = \{1\} \text{ and } SR = RS = 1, SS = R, RR = S \\
A &= \{T\} \text{ with } \overline{T} = T \\
R &= \{((S,T),(S,T),(S,T),(S,T),(S,T),(S,T),(S,T)), \\
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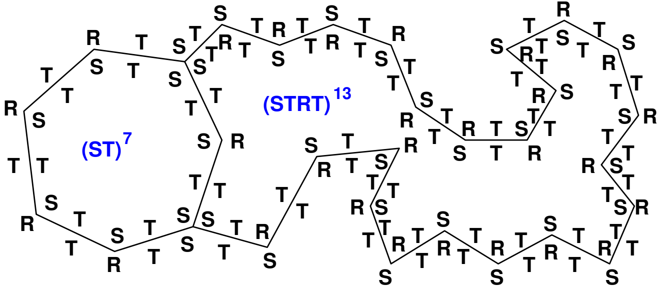

$$(ST)^7(TRTS)^{13} = (ST)^5(STST)(TRTS)(TRTS)^{12} = (ST)^5R(TRTS)^{12}$$

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You just have to chose the right pongo and edge alphabet!

Combinatorial Curvature

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Euler’s formula/genus condition

The total sum of our combinatorial curvature is always $+1$.

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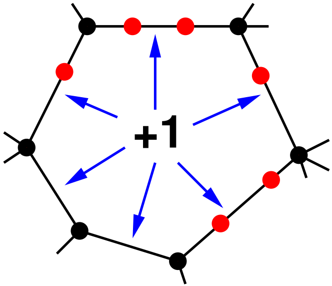
In Phase 1 we move the positive curvature to the half-edges:



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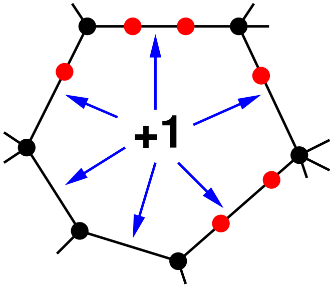

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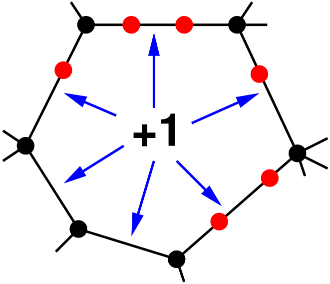

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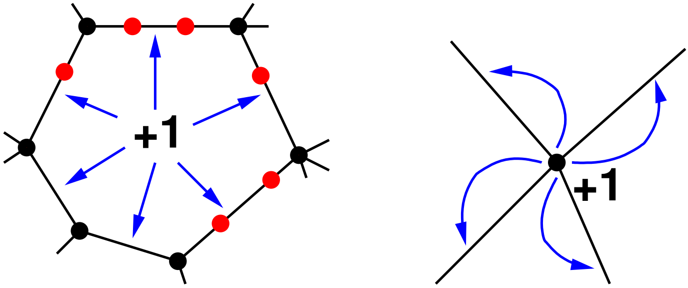

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Δ describes a step of the crawler, we sum curvature over $\langle\Delta\rangle$-orbits.

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If $S \geq 0$ then there is a $j \in L$ such that for all $i \in \mathbb{N}$ the partial sum

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a_i	2	-3	4	1	-5	3	2
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Corollary

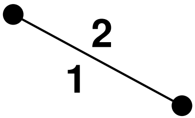
Assume that there are $k \in \mathbb{N}$ and $\varepsilon \leq 0$ such that for all $j \in L$ there is an $i \leq k$ with $s_{j,i} < \varepsilon$, then $S < \varepsilon \cdot \ell / k$.

Search for bad orbits of pubcrawl “EFEFE”

Data structure in computer

Id	E	F	F^{-1}	Rel
1	2			*
2	1			*
				*
				*
				*
				*

Illustration



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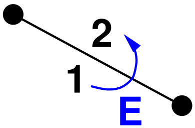

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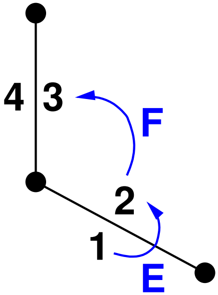

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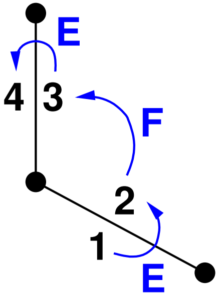

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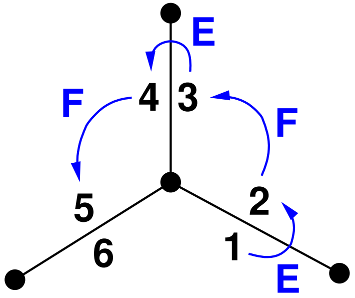

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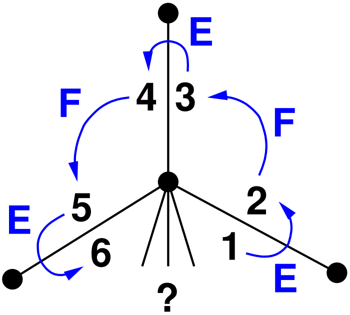

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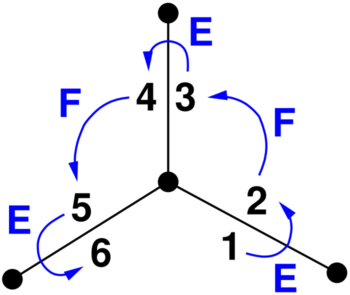

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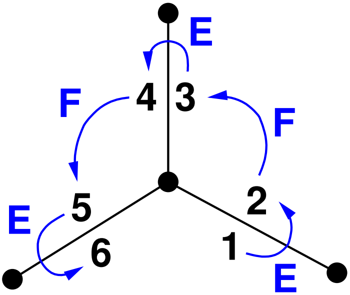

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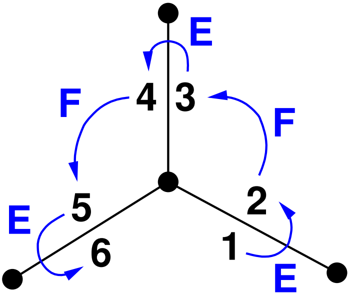

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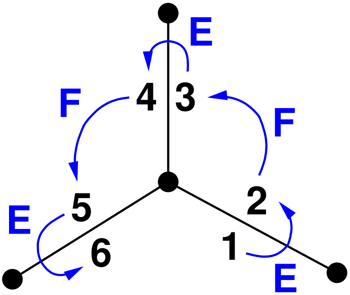

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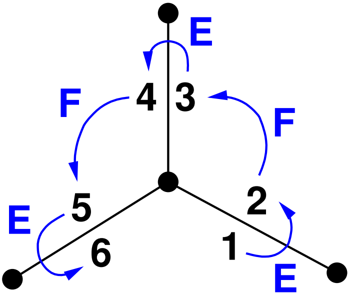

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Note that we use lower bounds for the vertex valencies!

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In the second case, the "Goes up and stays up" corollary tells us that

$\boxed{\text{the sum over every interior crawl orbit of length } \ell \text{ is } < \ell \cdot \varepsilon / k < 0.}$

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In the second case, the “Goes up and stays up” corollary tells us that

$\boxed{\text{the sum over every interior crawl orbit of length } \ell \text{ is } < \ell \cdot \varepsilon / k < 0.}$

Since the amount of positive curvature close to the boundary can be bounded from above by an expression in the boundary length, we get a

$\boxed{\text{linear isoperimetric inequality}}$

and thus have proved hyperbolicity. ■

What have we proved?

If this terminates, we have

- either found a bad cycle with non-negative curvature sum, or
- proved, that for every starting position in a crawl orbit

  $\boxed{\text{the partial sum after at most } k \text{ steps is } < \varepsilon}$

  for some global $k \in \mathbb{N}$ and some $\varepsilon < 0$.

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e.g.: $\langle S, T \mid S^3, T^2, (ST)^7, (STS^2T)^{13} \rangle$ is hyperbolic.

Outlook and plans

We want to

- investigate more ways of redistributing curvature.
- determine whether for every presentation of a hyperbolic group there is a successful curvature-redistribution scheme
  - easy for random presentations with low Gromov density.
- sort out details for a version for relative hyperbolicity.
- investigate applications to monoids and rewrite systems.
- find more interesting pongos — what do they do?
- generalise to "flat" jigsaw puzzles.
- develop further algorithms to solve the word problem, after proving the isoperimetric inequality.
- investigate lots of examples: send us your groups!