Aschbacher's Theorem

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Aschbacher's Theorem

Let $n \in \mathbb{N}$ and $\mathbb{F}_q$ the field with $q = p^e$ elements.
Let $V := \mathbb{F}_q^{1\times n}$ be the $\mathbb{F}_q$-vector space of row vectors.

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Theorem (Aschbacher 1984)

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ and $n \geq 2$. Then G lies in at least one of the classes $\mathcal{C}_1$ to $\mathcal{C}_9$ of subgroups of $\mathrm{GL}_n(\mathbb{F}_q)$.

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- I will show you a sketch of a proof of this statement.
- This is not the original formulation, which is more general.
- Alongside the sketch of the proof, we will
  - define $\mathcal{C}_1$ to $\mathcal{C}_9$, and
  - keep an eye on how one can find reduction homomorphisms computationally.

First analyse the natural module

The natural module $V = \mathbb{F}_q^{1\times n}$ could:

- have a *G*-invariant subspace (reducible),
- have a vector space structure over an extension field (semilinear),
- essentially have a vector space structure over a subfield (subfield).

Reducible: $\mathcal{C}_1$

G could lie in $\mathcal{C}_1$:

Definition of class $\mathcal{C}_1$: Reducible

$G \leq \mathrm{GL}_n(\mathbb{F}_q)$ lies in $\mathcal{C}_1$ if there is a subspace $0 < W < V$ with $Wg = W$ for all $g \in G$.

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We can decide computationally using the MeatAxe, whether such an invariant subspace W exists or not (see talk tomorrow).

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Assumption

From now on we assume that G acts irreducibly on V.

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Semilinear: $\mathcal{C}_3$

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$G \leq \mathrm{GL}_n(\mathbb{F}_q)$ lies in $\mathcal{C}_3$ if

- the natural module V is irreducible and
- there is a finite field $\mathbb{F}_{q^s}$, for which we can extend the $\mathbb{F}_q$-vector space structure of V to an $\mathbb{F}_{q^s}$-vector space structure of dimension n/s, such that:

$$\forall g \in G\ \exists \alpha_g \in \mathrm{Aut}(\mathbb{F}_{q^s}) \text{ with:}$$

$$(v + \lambda w) \cdot g = v \cdot g + \lambda^{\alpha_g} \cdot w \cdot g$$

for all $v, w \in V$ and all $\lambda \in \mathbb{F}_{q^s}$.

(i.e. the action of G on V is $\mathbb{F}_{q^s}$-semilinear)

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Non-absolutely irred. case: all automorphisms are trivial!

Subfield: $\mathcal{C}_5$

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$G \leq \mathrm{GL}_n(\mathbb{F}_q)$ lies in $\mathcal{C}_5$ if

- the natural module V is absolutely irreducible and
- there is a proper subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$ and $T \in \mathrm{GL}_n(\mathbb{F}_q)$ and $(\beta_g)_{g \in G}$ with $\beta_g \in \mathbb{F}_q$ such that

$$\beta_g \cdot T^{-1} g T \in \mathrm{GL}_n(\mathbb{F}_{q_0}) \text{ for all } g \in G.$$

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We can decide computationally whether G lies in $\mathcal{C}_5$ (see Glasby, Leedham-Green, and O'Brien (2006) and Carlson, N. and Roney-Dougal (2009)).

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Assumption

From now on we assume that G does not lie in $\mathcal{C}_5$.

Restrict to normal subgroup

Let $\overline{N} \lhd G/Z$ be minimal normal and $Z < N \leq G$ so that $N/Z = \overline{N}$.
($N = G$ is possible, but $G = Z$ not, since that would be reducible.)

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(Clifford's theorem shows that one of the above must hold.)

W not absolutely irreducible: $\mathcal{C}_3$

Remember: $Z < N \lhd G$ such that N/Z is minimal normal in G/Z.

Lemma

Let W be an irreducible submodule of $V|_N$. If W is not absolutely irreducible, then G lies in $\mathcal{C}_3$.

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Let now $\overline{N}$ be a minimal normal subgroup of G/Z and let $Z < N \lhd G$ be the full preimage.

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Theorem (Clifford)

The restriction $V|_N$ of the natural module to the normal subgroup N is a direct sum

$$V|_N = \bigoplus_{i=1}^{k} W_i$$

of irreducible N-modules W_i which are all G-conjugates of a single submodule $W \leq V|_N$, i.e. $W_i = Wg_i$ for some $g_i \in G$.

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Now we distinguish cases for this decomposition.

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Then G permutes the homogeneous components and lies in $\mathcal{C}_2$:

$$V|_N = \bigoplus_{i=1}^{k} W_i = \bigoplus_{j} \left(\bigoplus_{a} W_a^{(j)} \right)$$

where $W_a^{(j)} \cong W_b^{(\ell)}$ iff $j = \ell$.

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$G \leq \mathrm{GL}_n(\mathbb{F}_q)$ lies in $\mathcal{C}_2$ if

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Given the decomposition, we can compute the homomorphism $G \to S_m$.

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Assumption

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Analyse structure of minimal normal subgroup N/Z

Lemma (Minimal normal subgroups)

Let $1 < K \lhd H$ be a minimal normal subgroup. Then

$$K \cong T_1 \times T_2 \times \cdots \times T_k$$

and the T_i are copies of a simple group that are all conjugate under H.

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the T_i are pairwise isomorphic simple groups which are all conjugate under G/Z and thus G.

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We distinguish 3 cases:

1. the T_i are cyclic groups of prime order r (extraspecial)
2. the T_i are non-abelian simple and $k \geq 2$ (tensor-induced)
3. $k = 1$ and T_1 is non-abelian simple (almost simple)

Extraspecial: $\mathcal{C}_6$

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This class is in practice computationally under control.

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  - and G/N permutes the tensor factors transitively.

The Proof: The almost simple case

By now we have established that N/Z is a non-abelian simple group.

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G acts by conjugation on N/Z.

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We distinguish two cases:

- $\mathcal{C}_8$ (classical group in natural representation) and
- $\mathcal{C}_9$ (almost simple plus properties — "all that is left")

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- n is even, G is contained in $N_{\mathrm{GL}_n(\mathbb{F}_q)}(\mathrm{Sp}_n(\mathbb{F}_q))$ for some non-singular symplectic form, G/Z contains $\mathrm{PSp}_n(\mathbb{F}_q)$ and $(n, q) \notin \{(2, 2), (2, 3), (4, 2)\}$,
- q is a square, G is contained in $N_{\mathrm{GL}_n(\mathbb{F}_q)}(\mathrm{SU}_n(\mathbb{F}_{q^{1/2}}))$ for some non-singular Hermitian form, G/Z contains $\mathrm{PSU}_n(\mathbb{F}_{q^{1/2}})$ and $(n, q^{1/2}) \notin \{(2, 2), (2, 3), (3, 2)\}$,
- G is contained in $N_{\mathrm{GL}_n(\mathbb{F}_q)}(\Omega^{\epsilon}_n(\mathbb{F}_q))$, the corresponding $\mathrm{P}\Omega^{\epsilon}_n(\mathbb{F}_q)$ is simple and contained in G/Z.

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  - $n \geq 3$, and
  - q is odd if n is odd, and
  - ϵ is $-$ if $n = 4$, and
  - $(n, q) \notin \{(3, 3), (4, 2)\}$.

G/Z almost simple: $\mathcal{C}_9$

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This completes our proof of Aschbacher's Theorem. ∎

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The groups in classes $\mathcal{C}_8$ and $\mathcal{C}_9$ offer no opportunity to use geometric properties to find reduction homomorphisms.

They will have to be dealt with directly in constructive recognition.