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Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible:  $\mathcal{D}_1$ Not abs. irred:  $\mathcal{D}_3$ G/Z is simple:  $\mathcal{D}_6$  or  $\mathcal{D}_6$ Clifford theory W not abs. irred:  $\mathcal{D}_3$ V i<sub>N</sub> not homogeneous:  $\mathcal{D}_4$ V i<sub>N</sub> not homogeneous:  $\mathcal{D}_4$ V i<sub>N</sub> a cases for N/Z:  $\mathcal{D}_6$ - $\mathcal{D}_6$ 

Application

Classes D2 and I Class D7

# Aschbacher's Theorem revisited with a view to Constructive Matrix Group Recognition

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# Constructive recognition of matrix groups

#### Problem

Let  $\mathbb{F}_q$  be the field with q elements und

 $M_1,\ldots,M_k\in \operatorname{GL}_n(\mathbb{F}_q).$ 

Find for  $G := \langle M_1, \ldots, M_k \rangle$ :

- The group order |G| and
- an algorithm that, given  $M \in GL_n(\mathbb{F}_q)$ ,
  - decides, whether or not  $M \in G$ , and,
  - if so, expresses *M* as word in the *M<sub>i</sub>*.
- The runtime should be bounded from above by a polynomial in *n*, *k* and log *q*.
- A Monte Carlo Algorithm is enough. (Verification!)

If this problem is solved, we call  $\langle M_1, \ldots, M_k \rangle$  recognised constructively.

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# Reductions

Let  $G := \langle M_1, \ldots, M_k \rangle \leq \operatorname{GL}_n(\mathbb{F}_q)$ .

### A reduction is a group homomorphism

$$\begin{array}{rcl} \varphi & : & G & \to & H \\ & & M_i & \mapsto & P_i & \text{ for all } i \end{array}$$

### with the following properties:

- $\varphi(M)$  is explicitly computable for all  $M \in G$
- $\varphi$  is surjective:  $H = \langle P_1, \ldots, P_k \rangle$
- *H* is in some sense "smaller"
- or at least "easier to recognise constructively"
- e.g.  $H \leq S_m$  or  $H \leq \operatorname{GL}_{n'}(\mathbb{F}_{q'})$  with  $n' \log q' < n \log q$

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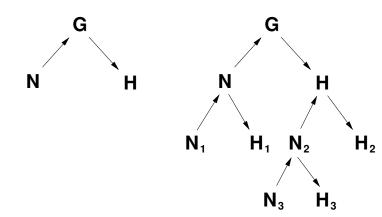
Aschbacher's Theorem

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#### Application

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## Recursive reduction: composition trees We get a tree:



Up arrows: inclusions Down arrows: homomorphisms Old idea, improvements are still being made

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# Which theorem of Aschbacher do I mean?

### Theorem (Aschbacher 1984)

Let  $G_0$  be a simple classical group over a finite field and  $G_0 \le G \le \operatorname{Aut}(G_0)$ . Let H < G such that  $HG_0 = G$ . Define geometrically classes  $\mathcal{C}_1$  to  $\mathcal{C}_8$  of subgroups of G. Then either H is a subgroup of at least one of the groups in classes  $\mathcal{C}_1$  to  $\mathcal{C}_8$ , or the following hold:

- There is a non-abelian simple group  $H_0$  with  $H_0 \le H \le \operatorname{Aut}(H_0)$ .
- The natural H-module V is absolutely irreducible.
- This representation for H cannot be realised over a smaller field.

### There is a number of simplifying lies on this slide!

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# A variant ...

Let  $n \in \mathbb{N}$  and  $\mathbb{F}_q$  the field with  $q = p^e$  elements. Let  $V := \mathbb{F}_q^{1 \times n}$  be the  $\mathbb{F}_q$ -vector space of row vectors.

### Theorem

Let  $G \leq GL_n(\mathbb{F}_q)$  and  $n \geq 2$ . Then G lies in at least one of the classes  $\mathfrak{D}_1$  to  $\mathfrak{D}_9$  of subgroups of  $GL_n(\mathbb{F}_q)$ .

- I will not tell you on this slide what the classes  $\mathcal{D}_1$  to  $\mathcal{D}_9$  are.
- I will show you a sketch of the proof of this statement.
- This is not new, lots of people have worked on this.
- Alongside the proof, we will
  - define  $\mathcal{D}_1$  to  $\mathcal{D}_9$ , and
  - keep an eye on how one can find reductions computationally.

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Application

Classes D<sub>2</sub> and D<sub>4</sub> Class D<sub>7</sub>

# Reducible: $\mathcal{D}_1$

## G could lie in $\mathcal{D}_1$ :

#### Definition of class $\mathcal{D}_1$

 $G \leq \operatorname{GL}_n(\mathbb{F}_q)$  lies in  $\mathcal{D}_1$  if there is a subspace 0 < W < V with Wg = W for all  $g \in G$ .

We can decide computationally using the MeatAxe, whether such an invariant subspace *W* exists or not.

### Assumption

From now on we assume that G acts irreducibly on V.

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Application

Classes D<sub>2</sub> and *L* Class D<sub>7</sub>

## Not absolutely irreducible: $\mathcal{D}_3$

*G* could act irreducibly but not absolutely irreducibly. (*G* acts absolutely irreducibly iff  $C_{GL_n(\mathbb{F}_q)}(G) = \{c \cdot 1\}$ .)

#### Lemma

If  $G \leq GL_n(\mathbb{F}_q)$  acts irreducibly but not absolutely irreducibly on the natural module V, then G lies in  $\mathcal{D}_3$ .

We can decide computationally using the MeatAxe, whether G acts absolutely irreducibly on V.

### Assumption

From now on we assume that G acts absolutely irreducibly on V.

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Application

Classes D2 and D4 Class D7

# Semilinear: $\mathcal{D}_3$

### Definition of class $\mathcal{D}_3$

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$  lies in  $\mathcal{D}_3$  if
  - the natural module V is irreducible and
  - there is a finite field 𝔽<sub>q<sup>s</sup></sub>, for which we can extend the 𝔽<sub>q</sub>-vector space structure of *V* to an 𝔽<sub>q<sup>s</sup></sub>-vector space structure of dimension *n*/*s*, such that:

 $\forall g \in G \ \exists \alpha_g \in Aut(\mathbb{F}_{q^s})$  with:

 $(\mathbf{v} + \lambda \mathbf{w}) \cdot \mathbf{g} = \mathbf{v} \cdot \mathbf{g} + \lambda^{\alpha_g} \cdot \mathbf{w} \cdot \mathbf{g}$ for all  $\mathbf{v}, \mathbf{w} \in V$  and all  $\lambda \in \mathbb{F}_{q^s}$ .

(i.e. the action of *G* on *V* is  $\mathbb{F}_{q^s}$ -semilinear)

Non-absolutely irred. case: all automorphisms are trivial!

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Application

Classes D<sub>2</sub> and D Class D<sub>7</sub> Subfield:  $\mathcal{D}_5$ 

*G* could lie in  $\mathcal{D}_5$ :

### Definition of class $\mathcal{D}_5$

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$  lies in  $\mathcal{D}_5$  if
  - the natural module V is absolutely irreducible and
  - there is a proper subfield  $\mathbb{F}_{q_0}$  of  $\mathbb{F}_q$  and  $T \in GL_n(\mathbb{F}_q)$ and  $(\beta_g)_{g \in G}$  with  $\beta_g \in \mathbb{F}_q$  such that

 $\beta_g \cdot T^{-1}gT \in \operatorname{GL}_n(\mathbb{F}_{q_0})$  for all  $g \in G$ .

We can decide computationally whether *G* lies in  $\mathcal{D}_5$  (see Glasby, Leedham-Green, and O'Brien (2006) and Carlson, N. and Roney-Dougal (submitted)).

### Assumption

From now on we assume that G does not lie in  $\mathcal{D}_5$ .

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Application Classes ற2 and ற4

## G/Z is simple: $\mathcal{D}_8$ or $\mathcal{D}_9$

From now on denote  $Z := Z(G) = G \cap Z(GL_n(\mathbb{F}_q))$ .

The group G/Z could be simple.

If G/Z were cyclic, then G would be abelian and V not absolutely irreducible.

Then G/Z is either a classical simple group in its natural representation (then *G* lies in  $\mathcal{D}_8$ ), or *G* lies in  $\mathcal{D}_9$ .

We cannot find a reduction in this case. Thus we have to recognise *G* constructively in some other way!

#### Assumption

Assume from now on that G/Z is not simple.

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Application

Classes D2 and D Class D7

# Classical in natural representation: $\mathcal{D}_8$

### Definition of class $\mathcal{D}_8$

 $G \leq GL_n(\mathbb{F}_q)$  lies in  $\mathcal{D}_8$  if G/Z contains a classical simple group in its natural representation in one of the following ways:

• G/Z contains  $PSL_n(\mathbb{F}_q)$  and  $(n, q) \notin \{(2, 2), (2, 3)\},\$ 

- *n* is even, *G* is contained in N<sub>GLn(𝔽q)</sub>(Sp<sub>n</sub>(𝔽q)) for some non-singular symplectic form, *G*/*Z* contains PSp<sub>n</sub>(𝔽q) and (*n*, *q*) ∉ {(2, 2), (2, 3), (4, 2)},
- q is a square, G is contained in  $N_{\operatorname{GL}_n(\mathbb{F}_q)}(\operatorname{SU}_n(\mathbb{F}_{q^{1/2}}))$ for some non-singular Hermitian form, G/Z contains  $\operatorname{PSU}_n(\mathbb{F}_{q^{1/2}})$  and  $(n, q^{1/2}) \notin \{(2, 2), (2, 3), (3, 2)\},$

• *G* is contained in  $N_{\operatorname{GL}_n(\mathbb{F}_q)}(\Omega_n^{\epsilon}(\mathbb{F}_q))$ , the corresponding  $P\Omega_n^{\epsilon}(\mathbb{F}_q)$  is simple and contained in *G*/*Z*. The group  $P\Omega_n^{\epsilon}(\mathbb{F}_q)$  is simple if and only if

- \*  $n \ge 3$ , and
- \* *q* is odd if *n* is odd, and
- \*  $\epsilon$  is if n = 4, and
- \*  $(n, q) \notin \{(3, 3), (4, 2)\}.$

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Application

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# G/Z almost simple: $\mathcal{D}_9$

### Definition of class $\mathcal{D}_9$

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$  lies in  $\mathcal{D}_9$ , if
  - it is not in  $\mathcal{D}_8$  and
  - there is a non-abelian simple group N and a group T with N ≤ T ≤ Aut(N) such that
    - $G/Z \cong T$  and
    - *V* gives rise to an absolutely irreducible projective representation for *T*,

which is not realisable over a proper subfield of  $\mathbb{F}_q$ .

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 $V|_N$  homogeneous:  $\mathcal{D}_4$ 3 cases for N/Z:  $\mathcal{D}_6-\mathcal{D}_9$ 

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# **Clifford theory**

Let now  $\overline{N}$  be a minimal normal subgroup of G/Z and let  $Z < N \triangleleft G$  be the full preimage.

### Theorem (Clifford)

The restriction  $V|_N$  of the natural module to the normal subgroup N is a direct sum

$$V|_N = \bigoplus_{i=1}^k W_i$$

of irreducible N-modules  $W_i$  which are all G-conjugates of a single submodule  $W \leq V|_N$ , i.e.  $W_i = Wg_i$  for some  $g_i \in G$ .

Now we distinguish cases for this decomposition.

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W not abs. irred.:  $\mathcal{D}_3$   $V|_N$  not homogeneous:  $\mathcal{D}_2$   $V|_N$  homogeneous:  $\mathcal{D}_4$ 3 cases for N/Z:  $\mathcal{D}_6-\mathcal{D}_9$ 

Application Classes D<sub>2</sub> and D<sub>4</sub>

# W not absolutely irreducible: $\mathcal{D}_3$

Remember:  $Z < N \triangleleft G$  such that N/Z is minimal normal.

#### Lemma

Let W be an irreducible submodule of  $V|_N$ . If W is not absolutely irreducible, then G lies in  $\mathcal{D}_3$ .

This is computationally under control, see "SMASH": Holt, Leedham-Green, O'Brien and Rees (1996) or Carlson, N., Roney-Dougal (submitted).

#### Assumption

From now on we assume that *W* is absolutely irreducible.

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## $V|_N$ not homogeneous: $\mathcal{D}_2$

Assume that not all  $W_i$  are isomorphic to W.

Then *G* permutes the homogeneous components and lies in  $\mathcal{D}_2$ :

$$V|_N = \bigoplus_{i=1}^k W_i = \bigoplus_j \left(\bigoplus_a W_a^{(j)}\right)$$

where 
$$W_a^{(j)} \cong W_b^{(l)}$$
 iff  $j = l$ .

#### Definition of class $\mathcal{D}_2$

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$  lies in  $\mathcal{D}_2$  if
  - the natural module V is absolutely irreducible and
  - there is Z < N ⊲ G such that V|<sub>N</sub> = ⊕<sup>k</sup><sub>i=1</sub> W<sub>i</sub> and the W<sub>i</sub> are absolutely irreducible F<sub>q</sub>N-modules and not all isomorphic.

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# $V|_N$ homogeneous: $\mathcal{D}_4$

Assume that all  $W_i$  are isomorphic to W and k > 1.

If dim<sub> $\mathbb{F}_q$ </sub>(*W*) = 1 then *N* would be scalar.

### Definition of class D<sub>4</sub>

- $G \leq GL_n(\mathbb{F}_q)$  lies in class  $\mathcal{D}_4$  if
  - the natural module V is absolutely irreducible and
  - there is  $N \triangleleft G$  such that  $V|_N = \bigoplus_{i=1}^k W_i$  with  $k \ge 2$ and  $W_i \cong W$  for all *i*, where *W* is absolutely irreducible  $\mathbb{F}_q N$ -module with  $\dim_{\mathbb{F}_q}(W) > 1$ .

#### Assumption

We assume from now on that  $W = V|_N$ .

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# Minimal normal subgroups

Now look at the group structure of N/Z:

Lemma (Minimal normal subgroups)

Let  $1 < K \triangleleft H$  be a minimal normal subgroup. Then

 $K \cong T_1 \times T_2 \times \cdots \times T_k$ 

and the  $T_i$  are copies of a simple group which are all conjugate under H.

Therefore,

$$N/Z \cong T_1 \times T_2 \times \cdots \times T_k,$$

the  $T_i$  are pairwise isomorphic simple groups which are all conjugate under G/Z and thus G.

We distinguish 3 cases:

- **()** the  $T_i$  are cyclic groups of prime order  $r(\mathcal{D}_6)$
- 2 the  $T_i$  are non-abelian simple and  $k \ge 2$  ( $\mathcal{D}_7$ )
- (3) k = 1 and  $T_1$  is non-abelian simple ( $\mathcal{D}_8$  or  $\mathcal{D}_9$ )

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# Extraspecial: D<sub>6</sub>

## Definition of class $\mathcal{D}_6$

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$  lies in  $\mathcal{D}_6$  if
  - the natural module V is absolutely irreducible,
  - $n = r^m$  for a prime r and
    - either *r* is odd and *G* has a normal subgroup *E* that is an extraspecial *r*-group of order  $r^{1+2m}$  and exponent *r*,
      - or r = 2 and G has a normal subgroup E that is either extraspecial of order  $2^{1+2m}$  or a central product of a cyclic group of order 4 with an extraspecial group of order  $2^{1+2m}$ ,
  - and in both cases the linear action of *G* on the  $\mathbb{F}_r$ -vector space E/Z(E) of dimension 2m is irreducible.

This class is in practice computationally under control.

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# Tensor-induced: $\mathcal{D}_7$

### Definition of class $\mathcal{D}_7$

- $G \leq \operatorname{GL}_n(\mathbb{F}_q)$  lies in  $\mathcal{D}_7$  if
  - the natural module V is absolutely irreducible and,
  - there is  $Z < N \triangleleft G$  such that for some k > 1,

$$N \cong \underbrace{T \circ \cdots \circ T}_{k \text{ factors}} \quad \text{(central product)},$$

where T/Z is a non-abelian simple group, such that:

- V|<sub>N</sub> ≅ W<sub>1</sub> ⊗<sub>F<sub>q</sub></sub> ··· ⊗<sub>F<sub>q</sub></sub> W<sub>k</sub> where the W<sub>i</sub> are absolutely irreducible F<sub>q</sub>T-modules of the same dimension on which Z acts as scalars,
- and *G*/*N* permutes the tensor factors transitively.

Max Neunhöffer

Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible:  $\mathfrak{D}_1$ Not abs. irred:  $\mathfrak{D}_3$ Subfield:  $\mathfrak{D}_2$ Gilfiord theory Win ot abs. irred:  $\mathfrak{D}_3$ Vi<sub>W</sub> not thomogeneous:  $\mathfrak{D}_4$ Vi<sub>W</sub> not thomogeneous:  $\mathfrak{D}_4$ 3 cases for NZ:  $\mathfrak{D}_6$ - $\mathfrak{D}_9$ 

Application Classes D<sub>2</sub> and D<sub>4</sub> Class D<sub>7</sub> Finding reductions for groups in  $\mathcal{D}_2$  and  $\mathcal{D}_4$ 

 $\mathcal{D}_2$  and  $\mathcal{D}_4$  in this formulation have in common:

- In both cases there is an N with  $Z < N \triangleleft G$ .
- $V|_N$  is reducible such that the MeatAxe can:
  - determine whether  $H \leq N$  for some  $H \triangleleft G$  and
  - find a reduction in that case.

Since we can compute normal closures in *G*, all we need is to solve:

#### Problem

Find one element  $n \in N \setminus Z$  with high probability.

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Background Constructive recognition Reductions Composition trees

Aschbacher's Theorem

A variant for GL and its proof The Statement Reducible:  $D_1$ Not abs. irred.:  $D_3$ Subfield:  $D_2$ G/Z is simple:  $D_3$  or  $D_5$ Cillford theory W not abs. irred.:  $D_3$ Vi<sub>N</sub> not homogeneous:  $D_4$ 2 access for N/Z:  $D_6$ — $D_9$ 

Application Classes D<sub>2</sub> and D<sub>4</sub> Class D<sub>7</sub>

# Finding a reduction for groups in $\mathcal{D}_7$

Also the definition of  $\mathcal{D}_7$  involves *N* with  $Z < N \triangleleft G$ .

However, this time  $V|_N$  is irreducible, so we do not notice, whether some  $H \le N!$ 

But: N in  $\mathcal{D}_7$  lies itself in  $\mathcal{D}_4$ !

#### Idea

If we had a provably nice way to produce elements in a normal subgroup, then we could use the trick twice.